EE113: DIGITAL SIGNAL PROCESSING

Midterm 1 Practice Problems

Problem 1

Compute the convolution of $x[n]$ and $h[n]$, $y[n] = x[n] * h[n]$.

a) $x[n] = \{\underset{n=0}{\underset{\uparrow}{-3}}, 7, 4\}$ and $h[n] = \{\underset{n=0}{\underset{\uparrow}{4}}, 3, 2, 1\}$

b) $x[n] = u[n] - u[n-7]$ and $h[n] = (0.9)^n u[n]$

Solution:

a) $y[n] = \{\underset{n=0}{\underset{\uparrow}{-12}}, 19, 31, 23, 15, 4\}$

b) We can express the convolution as

$$y[n] = \sum_{k=-\infty}^{\infty} 0.9^k u[k](u[n-k] - u[n-7-k])$$

We can omit the $u[k]$ term and reduce the sum to

$$y[n] = \sum_{k=0}^{\infty} 0.9^k (u[n-k] - u[n-7-k])$$

Notice that for $n < 0$, both $u[n-k]$ and $u[n-7-k]$ are 0 for every k in the range of the sum. Therefore, $y[n] = 0$ for $n < 0$.

For $0 \leq n < 7$, $u[n-7-k]$ is 0 for every k in the range of the sum, so $y[n]$ is:

$$y[n] = \sum_{k=0}^{\infty} 0.9^k (u[n-k])$$

$$y[n] = \sum_{k=0}^{n} 0.9^k = \frac{1-0.9^{n+1}}{1-0.9}$$

For $n \geq 7$:

$$y[n] = \sum_{k=0}^{\infty} 0.9^k (u[n-k] - u[n-7-k])$$

$$y[n] = \sum_{k=n-6}^{n} 0.9^k$$

We can make a substitution $r = n - k$, then

$$y[n] = \sum_{r=0}^{6} 0.9^{n-r} = 0.9^n \frac{1 - 0.9^{-7}}{1 - 0.9^{-1}}$$

Finally,

$$y[n] = \frac{1 - 0.9^{n+1}}{1 - 0.9}(u[n] - u[n-7]) + 0.9^n \frac{1 - 0.9^{-7}}{1 - 0.9^{-1}} u[n-7]$$

Problem 2

a) Consider a discrete-time complex signal $x[n] = A[n]e^{j\phi[n]}$, where $A[n] = |x[n]|$ and $\phi[n]$ is the phase of the signal x[n].
Derive the relationship between $A[n]$ and $A[-n]$, and $\phi[n]$ and $\phi[-n]$ when the signal is

  i) conjugate symmetric

  ii) conjugate antisymmetric

b) Now consider a discrete-time complex signal $x[n] = a[n] + jb[n]$, where $a[n]$ is the real part and $b[n]$ is the imaginary part of $x[n]$. Is $a[n]$ and $b[n]$ odd or even when the signal x[n] is:

  i) conjugate symmetric

  ii) conjugate antisymmetric

Solution:

a) i) The condition for conjugate symmetric signals is

$$x^*[n] = x[-n]$$

In our case,

$$A[n]e^{-j\phi[n]} = A[-n]e^{j\phi[-n]}$$

Hence,

$$A[n] = A[-n]$$
$$-\phi[n] = \phi[-n]$$

ii) The condition for conjugate antisymmetric signals is

$$x^*[n] = -x[-n]$$

In our case,

$$A[n]e^{-j\phi[n]} = -A[-n]e^{j\phi[-n]}$$
$$A[n]e^{-j\phi[n]} = A[-n]e^{\pm j\pi}e^{j\phi[-n]}$$

$$A[n]e^{-j\phi[n]} = A[-n]e^{j\phi[-n]\pm\pi}$$

Hence,

$$A[n] = A[-n]$$

$$-\phi[n] = \phi[-n] \pm \pi$$

b) i) The condition for conjugate symmetric signals is

$$x^*[n] = x[-n]$$

In our case,

$$a[n] - jb[n] = a[-n] + jb[-n]$$

For the equality to be true, $a[n] = a[-n]$ and $-b[n] = b[-n]$. Hence, $a[n]$ is an even signal and $b[n]$ is an odd signal.

ii) The condition for conjugate antisymmetric signals is

$$x^*[n] = -x[-n]$$

In our case,

$$a[n] - jb[n] = -a[-n] - jb[-n]$$

Hence, $a[n]$ is an odd signal and $b[n]$ is an even signal.

Problem 3

Assume x[n] has nonzero samples only in the interval $-N_1 \leq n \leq N_2$. Generally, over what interval of time will the following sequence have non-zero samples:

$$y[n] = x[n] * x[n]$$

Solution: *From the property of convolution derived in the problem 3.23 and covered in Discussion 4, we know that if two signals $x[n]$ and $h[n]$ are such that $x[n]$ only has non-zero samples in the range $N_{x1} \leq n \leq N_{x2}$ and $h[n]$ has non-zero samples only in the range $N_{h1} \leq n \leq N_{h2}$, then their convolution $y[n] = x[n] * h[n]$ can only have non-zero samples in the range $N_{h1} + N_{x1} \leq n \leq N_{h2} + N_{x2}$*

For $y[n] = x[n] * x[n]$, $y[n]$ will generally have non-zero samples in the range $-2N_1 \leq n \leq 2N_2$.

Problem 4

Prove the distributive property of the periodic convolution:

$$\tilde{x}[n] \otimes (\tilde{y}[n] + \tilde{z}[n]) = \tilde{x}[n] \otimes \tilde{y}[n] + \tilde{x}[n] \otimes \tilde{z}[n]$$

Solution: The periodic convolution of two signals $\tilde{h}[n]$ and $\tilde{x}[n]$ is defined as $\tilde{x}[n]\otimes\tilde{h}[n] = \sum_{k=0}^{N-1} \tilde{x}[k]\tilde{h}[n-k]$, where $\otimes$ is the periodic convolution operator and N is the period of both $\tilde{x}[n]$ and $\tilde{h}[n]$.

$$\tilde{x}[n] \otimes (\tilde{y}[n] + \tilde{z}[n]) = \sum_{k=0}^{N-1} \tilde{x}[k](\tilde{y}[n-k] + \tilde{z}[n-k])$$

$$= \sum_{k=0}^{N-1} \tilde{x}[k]\tilde{y}[n-k] + \sum_{k=0}^{N-1} \tilde{x}[k]\tilde{z}[n-k]$$

$$= \tilde{x}[n] \otimes \tilde{y}[n] + \tilde{x}[n] \otimes \tilde{z}[n]$$

Problem 5

Consider the following system: $y[n] = \sum_{k=0}^{n} \frac{1}{2^k} x[k]$

(a) Is the system linear? Prove your answer.

(b) Is the system time-invariant? Prove your answer.

(c) Is the system causal? Prove your answer.

(d) Is the system BIBO stable? Prove your answer. (*Hint: You may need to use triangle inequality:* $|x + y| \leq |x| + |y|$)

Solution:

(a) The system is linear.

$$Sys\{\alpha_1 x_1[n] + \alpha_2 x_2[n]\} = \sum_{k=1}^{n} \frac{1}{2^k}(\alpha_1 x_1[k] + (\alpha_2 x_2[k])$$

$$= \sum_{k=0}^{n} \alpha_1 \frac{1}{2^k} x_1[k] + \sum_{k=0}^{n} \alpha_2 \frac{1}{2^k} x_2[k]$$

$$= \alpha_1 \sum_{k=0}^{n} \frac{1}{2^k} x_1[k] + \alpha_2 \sum_{k=0}^{n} \frac{1}{2^k} x_2[k]$$

$$= \alpha_1 Sys\{\alpha_1 x_1[n]\} + \alpha_2 \{\alpha_2 x_2[n]\}$$

(b) The system is time-varying.

$$Sys\{x[n-m]\} = \sum_{k=0}^{n} \frac{1}{2^k} x[k-m]$$

$$y[n-m] = \sum_{k=0}^{n-m} \frac{1}{2^k} x[k]$$

The two sums are not equal.

(c) The system is causal since the output at time n, only depends on the samples from time n and before.

(d) The system is BIBO stable. Let $x[n]$ be bounded: $|x[n]| \leq B_x$.

$$|y[n]| = \left| \sum_{k=0}^{n} \frac{1}{2^k} x[k] \right|$$

Using triangle inequality

$$|y[n]| \leq \sum_{k=0}^{n} \left| \frac{1}{2^k} x[k] \right| = \sum_{k=0}^{n} \left| \frac{1}{2^k} \right| |x[k]|$$

The sum $\sum_{k=1}^{n} \left| \frac{1}{2^k} \right| |x[k]|$ is certainly smaller than

$$\sum_{k=0}^{n} \left| \frac{1}{2^k} \right| B_x$$

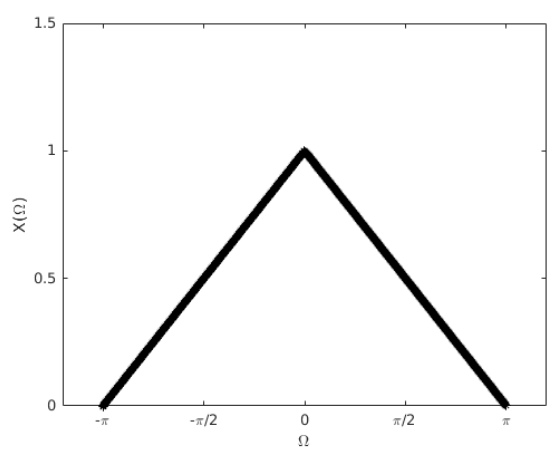
Hence,

$$|y[n]| \leq \sum_{k=0}^{n} \left| \frac{1}{2^k} \right| B_x \leq B_x \sum_{k=0}^{n} \left| \frac{1}{2^k} \right| = B_x \frac{1 - (1/2)^{n+1}}{1 - 1/2}$$

so $y[n]$ is bounded.

Problem 6

Consider a signal $x[n]$ that has a DTFT depicted in the figure below in the range $[-\pi, \pi]$.



Find the expression for the DTFT of the signals below:

(a) $x_1[n] = nx[n-1]$

(b) $x_2[n] = e^{j\frac{\pi n}{2}}(x[n] * x[n])$

(b) $x_o[n]$, the odd part of $x[n]$

Solution:

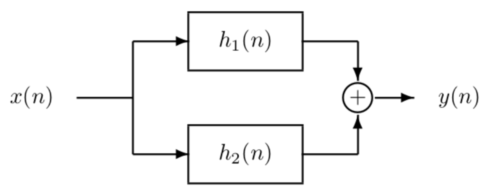
(a) $X_1(\Omega) = \frac{d}{d\Omega}(X(\Omega)e^{-j\Omega})$

(b) $X_2(\Omega) = X^2(\Omega - \pi/2)$

(b) $X_o(\Omega) = \frac{X(\Omega)-X(-\Omega))}{2}$

Problem 7

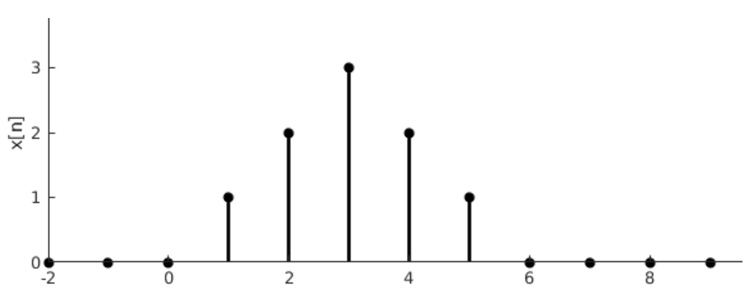
Consider the system composed of parallel connection of two LTI systems.



(a) If unit-step response of the equivalent system (the response when the input is a unit-step function) is $y[n] = r[n+1] - r[n-1]$ and $h_1[n] = u[n] - 2u[n-1] + u[n-2]$, find and sketch $h_2[n]$.

(b) Find the equivalent impulse response of the system $h_{eq}[n]$. The equivalent response is defined by the following relation: $y[n] = x[n] * h_{eq}[n]$.

(c) Find the response of the system $y[n]$ for $x[n]$ shown in the figure below.



Solution:

(a) The system response to $u[n]$ is

$$y[n] = u[n] * (h_1[n] + h_2[n]) = u[n] * h_1[n] + u[n] * h_2[n] \text{ (1 pt)}$$

Furthermore,

$$\begin{aligned}u[n] * h_1[n] &= u[n] * (u[n] - 2u[n-1] + u[n-2]) \\ &= u[n] * u[n] - 2u[n-1] * u[n] + u[n-2] * u[n] \\ &= r[n+1] - 2r[n] + r[n-1] \text{ (2 pts)}\end{aligned}$$

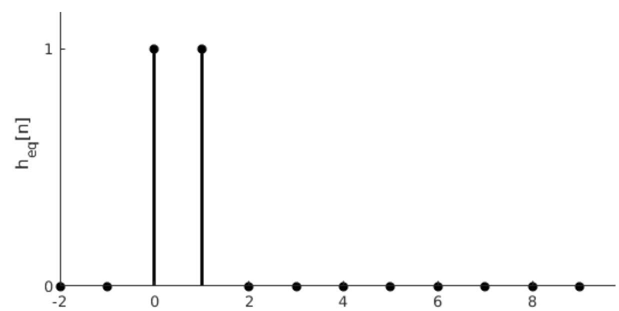
Where we have used the fact that $u[n] * u[n] = r[n+1]$. Now we can find $h_2[n]$,

$$u[n] * h_2[n] = y[n] - u[n] * h_1[n]$$

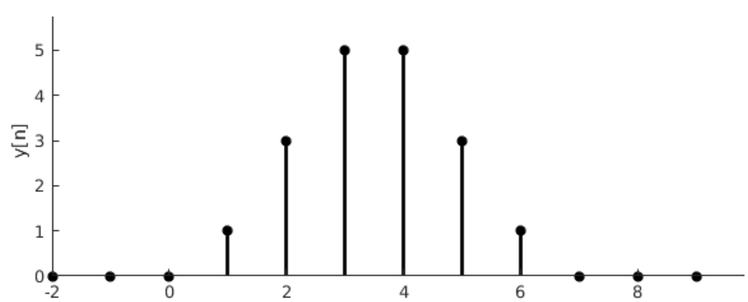
$$u[n] * h_2[n] = r[n+1] - r[n-1] - r[n+1] + 2r[n] - r[n-1]$$
$$u[n] * h_2[n] = 2r[n] - 2r[n-1]$$
$$u[n] * h_2[n] = u[n] * 2u[n-1] - u[n] * 2u[n-2]$$
$$u[n] * h_2[n] = u[n] * (2u[n-1] - 2u[n-2])$$

Hence, $h_2[n] = 2u[n-1] - 2u[n-2] = 2\delta[n-1]$. (1 pt)

(b) Since this is an addition of the outputs of two parallel systems, $h_{eq} = h_1[n] + h_2[n] = u[n] - 2u[n-1] + u[n-2] + 2u[n-1] - 2u[n-2] = u[n] - u[n-2]$.



(c) The response of the system is shown.



Problem 8

Let $\tilde{x}[n]$ be a periodic signal with period N. Its DTFS representation is given by

$$\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn},$$

where $\tilde{c}_k$ are the DTFS coefficients.

Show that if $\tilde{x}[n]$ is a complex signal and conjugate symmetric ($\tilde{x}^*[n] = \tilde{x}[-n]$), then $\text{Im}\{\tilde{c}_k\} = 0$.

Solution 1:

$$\tilde{c}_k = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

We take the conjugate of both sides of the expression above

$$\tilde{c}_k^* = \sum_{n=0}^{N-1} x^*[n]e^{j\frac{2\pi}{N}kn}$$

Using the fact that $\tilde{x}^*[n] = \tilde{x}[-n]$

$$\tilde{c}_k^* = \sum_{n=0}^{N-1} x[-n]e^{j\frac{2\pi}{N}kn}$$

We make the substitution $m = -n$

$$\tilde{c}_k^* = \sum_{m=1-N}^{0} x[m]e^{-j\frac{2\pi}{N}km}$$

We can freely shift the starting index of the summation above without changing its value and we shift it by $N-1$ to the right

$$\tilde{c}_k^* = \sum_{m=0}^{N-1} x[m]e^{-j\frac{2\pi}{N}km}$$

The expressions for $\tilde{c}_k^*$ and $\tilde{c}_k$ are identical, therefore $c_k^* = c_k$. This implies that $\text{Im}\{\tilde{c}_k\} = 0$.

Solution 2:

$$\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$$

$$\tilde{x}^*[n] = \sum_{k=0}^{N-1} \tilde{c}_k^* e^{-j\frac{2\pi}{N}kn}$$

$$\tilde{x}[-n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{-j\frac{2\pi}{N}kn}$$

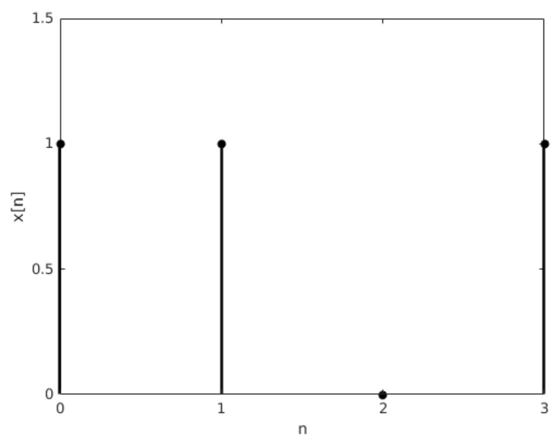
$$\tilde{x}[-n] - \tilde{x}^*[n] = \sum_{k=0}^{N-1} (\tilde{c}_k - \tilde{c}_k^*) e^{-j\frac{2\pi}{N}kn} = 0$$

$$2\sum_{k=0}^{N-1} \text{Im}\{\tilde{c}_k\} e^{-j\frac{2\pi}{N}kn} = 0$$

This implies that either $\text{Im}\{\tilde{c}_k\}$ is 0 or orthogonal to $e^{-j\frac{2\pi}{N}kn}$. However, since $e^{-j\frac{2\pi}{N}kn}$ is an orthogonal basis, its null space is $\{\emptyset\}$. therefore $\text{Im}\{\tilde{c}_k\} = 0$.

Problem 9

Consider a periodic signal $\tilde{x}[n]$ signal with one if its periods shown in the figure below. Calculate its DTFS coefficients $\tilde{c}_k$.



Solution: After applying the DTFS analysis equation on $x[n]$, we get

$$\tilde{c}_k = \frac{1}{4} + \frac{1}{4}e^{-j\frac{2\pi}{4}k} + \frac{1}{4}e^{-j\frac{2\pi}{4}3k}$$

This evaluates to

$$\tilde{c}_0 = 0.75$$

$$\tilde{c}_1 = 0.25$$

$$\tilde{c}_2 = -0.25$$

$$\tilde{c}_3 = 0.25$$