EE113 - Fall 2012 - Quiz 2 - Solution

1. Consider the constant-coefficient difference equation:

$$y(n) - 0.5y(n-1) = u(n), y(-1) = 2, n \geq 0$$

A. Find $y(0)$ and $y(\infty)$ using the initial and final value theorems.

B. Find the step-response of the system by relying only on the unilateral z-transform.

Solution:

A. Take the unilateral z-transform:

$$Y^+(z) - 0.5(z^{-1}Y^+(z) + y(-1)) = \frac{z}{z-1}$$

Solving for $Y^+(z)$,

$$Y^+(z) = \frac{2z}{z-1}$$

Therefore, using the initial and final value theorems, we have

$$\begin{aligned} y(0) &= \lim_{z\to\infty} Y^+(z) = 2 \\ y(\infty) &= \lim_{z\to 1}(z-1)Y^+(z) = \lim_{z\to 1} 2z = 2 \end{aligned}$$

B. From part (**A**), we have

$$Y^+(z) = \frac{2z}{z-1}$$

which implies that

$$y^+(n) = 2u(n)$$

So $y(n) = 2u(n+1)$.

2. Let $X(\omega)$ denote the DTFT of the signal $x(n)$ shown in Fig. 1. Perform the following without explicitly evaluating $X(\omega)$.

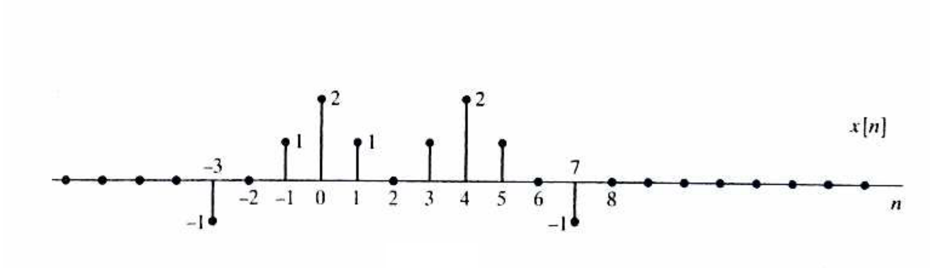

Figure 1: Figure for Problem 2.

A. Evaluate $X(0)$:

B. Evaluate $\theta(\omega)$.

C. Evaluate $\int_{-\pi}^{\pi} X(\omega) d\omega$

Solution:

A.

$$\begin{aligned} X(0) &= \left. \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right|_{\omega=0} \\ &= \sum_{n=-\infty}^{\infty} x(n) \\ &= 6 \end{aligned}$$

B. Define $x(n) = x_0(n-2)$ where $x_0(n)$ is given by Fig. 2:

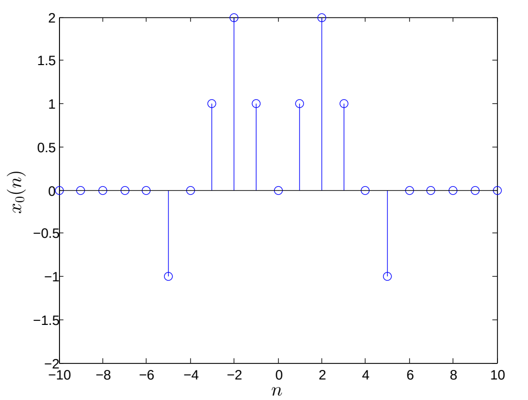

Figure 2: Sequence plot of $x_0(n)$.

Since $x_0(n)$ is symmetric over y-axis, the phase $\theta_0(\omega)$ of $X_0(\omega)$ is zero: $\theta_0(\omega) = 0$. By shifting property, we have that $X(\omega) = X_0(\omega)e^{-j2\omega} = |X_0(\omega)|e^{j\theta_0(\omega)-j2\omega}$. The phase of $X(\omega)$ is therefore $\theta(\omega) = -2\omega$.

C. $\int_{-\pi}^{\pi} X(\omega) d\omega = 2\pi x(0) = 4\pi$

3. For the system shown in Fig. 3, determine the impulse response if $H(\omega)$ is an ideal lowpass filter, passing frequencies less than $\pi/2$. That is, $H(\omega) = 1$ for $|\omega| \leq \pi/2$ and is zero otherwise.

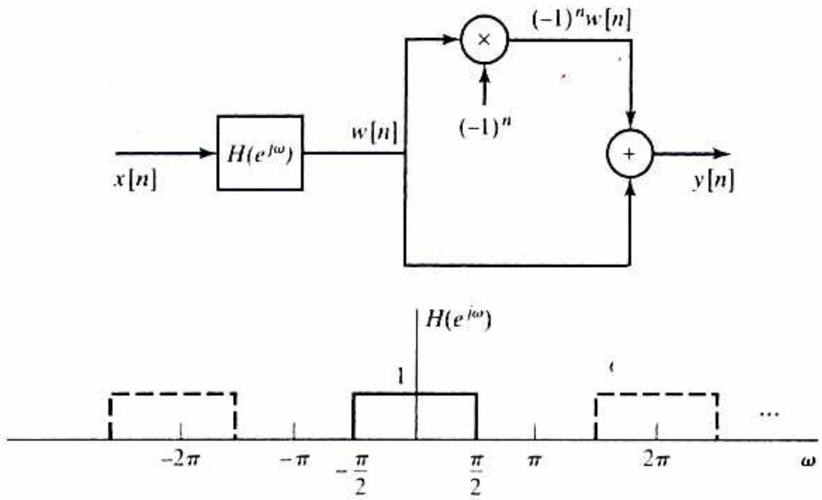

Figure 3: Figure for Problem 3.

Solution: When we input $x(n) = \delta(n)$, we have that $w(n) = h(n)$, which is the impulse response of the low-pass filter. Therefore, the DTFT of $w(n)$ is $H(e^{j\omega})$. Finally, the output $y(n) = w(n) + (-1)^n w(n) = h(n) + (-1)^n h(n)$. The DTFT of $y(n)$ can be found as:

$$Y(e^{j\omega}) = H(e^{j\omega}) + H(e^{j(\omega-\pi)})$$

Since $(-1)^n w(n)$ can be written as $e^{j\pi n} w(n)$ and we used the DTFT property:

$$e^{j\omega_0 n} x(n) \Longleftrightarrow X(e^{j(\omega-\omega_0)})$$

Now, we can write $H(e^{j\omega})$ as:

$$H(e^{j\omega}) = \begin{cases} 1, & -\frac{\pi}{2} \leq |\omega| \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

and therefore $H(e^{j(\omega-\pi)})$ can be written as:

$$H(e^{j(\omega-\pi)}) = \begin{cases} 1, & \frac{\pi}{2} \leq \omega \leq \pi \\ 1, & -\pi \leq \omega \leq -\frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

Therefore, $Y(e^{j\omega}) = 1$ for all ω. The inverse DTFT is hence:

$$y(n) = \delta(n)$$