1. (a) The difference equation that describes the system is

$$y(n) + 3y(n-1) = x(n) + 2x(n-1)$$

(b) To find the impulse response, we need to find the system response to the input $x(n) = \delta(n)$. In this case, the output $y(n) = h(n)$ is the impulse response sequence. This is equivalent to solving the difference equation.

$$h(n) + 3h(n-1) = \delta(n) + 2\delta(n-1), \qquad h(-1) = 0$$

This equation becomes homogeneous for $n \geq 2$ hence it is described by the homogeneous equation

$$h(n) + 3h(n-1) = 0$$

The characteristic equation associated with this system is

$$\lambda + 3 = 0$$

which has a single root at $\lambda = -3$. Then $h(n)$ is given by

$$h(n) = C\,(-3)^n\,, \qquad n \geq 2$$

To find C, we need to find $h(1)$ as follows

$$\begin{aligned} h(0) &= -3h(-1) + \delta(0) + 2\delta(-1) = 1 \\ h(1) &= -3h(0) + \delta(1) + 2\delta(0) = -1 \end{aligned}$$

Substituting by $h(1) = -1$, we find that $C = 1/3$ and $h(n)$ is now given by

$$h(n) = \frac{1}{3}\,(-3)^n\,, \qquad n \geq 1$$

We know that the system is causal, then $h(n) = 0$ for $n < 0$. We also found that $h(0) = 1$ then $h(n)$ is

$$h(n) = \delta(n) + \frac{1}{3}\,(-3)^n\,u(n-1)$$

(c) Take the z-transform of the difference equation that describes the system, we get

$$Y(z) + 3z^{-1}Y(z) = X(z) + 2z^{-1}X(z)$$

then

$$Y(z)\left(1+3z^{-1}\right) = X(z)\left(1+2z^{-1}\right)$$

and the transfer function $H(z)$ is given by

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{1+2z^{-1}}{1+3z^{-1}} \\ &= \frac{1+(3z^{-1}-z^{-1})}{1+3z^{-1}} = 1 - \frac{z^{-1}}{1+3z^{-1}} \end{aligned}$$

Since the system is causal, the region of convergence associated with this equation is $|z| > 3$. Using the following z-transform pairs

$$\begin{aligned} \delta(n) &\leftrightarrow 1 \\ (a)^n u(n) &\leftrightarrow \frac{1}{1-az^{-1}} \end{aligned}$$

and the following property,

$$x(n-1) \leftrightarrow z^{-1}X(z)$$

We find the impulse response sequence $h(n)$ to be

$$h(n) = \delta(n) - (-3)^{n-1}\, u(n-1)$$

(d) The system is relaxed and described be a constant coefficient difference equation, therefore, LTI. We can use find the output sequence by convolving the input sequence with the impulse response sequence.

$$\begin{aligned} y(n) &= h(n) \star x(n) = \left[\delta(n) - (-3)^{n-1}\, u(n-1)\right] \star u(-n-1) \\ &= \underbrace{\delta(n) \star u(-n-1)}_{y_1(n)} - \underbrace{(-3)^{n-1}\, u(n-1) \star u(-n-1)}_{y_2(n)} \end{aligned}$$

We evaluate $y_1(n)$ and $y_2(n)$ as follows

$$y_1(n) = \delta(n) \star u(-n-1) = u(-n-1)$$

and

$$
\begin{aligned}
y_2(n) &= (-3)^{n-1}\, u(n-1) \star u(-n-1) \\
&= \sum_{k=-\infty}^{\infty} (-3)^{k-1}\, u(k-1)u(-n+k-1) \\
&= \begin{cases} \sum_{k=n+1}^{\infty} (-3)^{k-1} & n \geq 0 \\ \sum_{k=1}^{\infty} (-3)^{k-1} & n < 0 \end{cases} \\
&= \begin{cases} (-3)^n \sum_{k=0}^{\infty} (-3)^{k} & n \geq 0 \\ \sum_{k=0}^{\infty} (-3)^{-k} & n < 0 \end{cases} \\
&= \begin{cases} (-3)^n \dfrac{1}{1-(-3)} & n \geq 0 \\ \dfrac{1}{1-(-1/3)} & n < 0 \end{cases} \\
&= \frac{1}{4}(-3)^n u(n) + \frac{1}{4} u(-n-1)
\end{aligned}
$$

Then $y(n)$ is given by

$$
\begin{aligned}
y(n) &= y_1(n) - y_2(n) \\
&= \frac{3}{4} u(-n-1) - \frac{1}{4}(-3)^n u(n)
\end{aligned}
$$

(e) The system is not BIBO stable since it is a causal system and the impulse response sequence has an unstable mode $\lambda = -3$ which means that $\sum_{n=-\infty}^{\infty} h(n) \rightarrow \infty$.

2. (a) We first solve for the impulse response assuming the system is relaxed. As $x(n) = \delta(n)$, we have

$$\begin{aligned} h(n) &= 0; \quad n < 0 \\ h(0) &= 1; \\ h(1) &= \frac{1}{3}; \\ h(n) - \frac{4}{3}h(n-1) + \frac{1}{3}h(n-2) &= 0; \quad n \geq 2 \end{aligned}$$

The characteristic equation for the homogeneous equation (when $n \geq 2$) is:

$$\begin{aligned} \lambda^2 - \frac{4}{3}\lambda + \frac{1}{3} &= 0 \\ (\lambda - \frac{1}{3})(\lambda - 1) &= 0 \end{aligned}$$

$$\lambda_1 = \frac{1}{3}, \quad \lambda_2 = 1$$

There are two distinct modes, so the general solution for the homogeneous equation is:

$$h(n) = C_1 \left(\frac{1}{3}\right)^n + C_2(1)^n = C_1 \left(\frac{1}{3}\right)^n + C_2; \quad n \geq 2$$

Substitute $h(n)$ to the initial conditions $(n = 0, 1)$.

$$\begin{aligned} C_1 + C_2 &= 1 \\ \frac{C_1}{3} + C_2 &= \frac{1}{3} \end{aligned}$$

$$C_1 = 1, \quad C_2 = 0$$

Therefore, the impulse response of the relaxed system is:

$$h(n) = \left(\frac{1}{3}\right)^n u(n)$$

(b) As the impulse response $h(n)$ has been computed in part (a),we use zero-state and zero-input method to find the complete solution. Given $x(n) = u(n)$, the zero-state response is:

$$\begin{aligned}
y_{zs}(n) &= h(n) \star u(n) \\
&= \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^k u(k)u(n-k) \\
&= \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k u(n) \\
&= \frac{1-\left(\frac{1}{3}\right)^{n+1}}{1-\frac{1}{3}} u(n) \\
&= \frac{1}{2}\left[3-\left(\frac{1}{3}\right)^n\right] u(n)
\end{aligned}$$

The general solution for the zero-input response is the same as the solution for homogeneous equation.

$$y_{zi} = C_1 \left(\frac{1}{3}\right)^n + C_2; \quad \mathit{for\ any}\ n$$

Apply the original initial conditions at $n = -1$ and -2.

$$\begin{aligned}
3C_1 + C_2 &= 1 \\
9C_1 + C_2 &= -2
\end{aligned}$$

$$C_1 = -\frac{1}{2}, \quad C_2 = \frac{5}{2}$$

The zero-input response is:

$$y_{zi} = -\frac{1}{2}\left(\frac{1}{3}\right)^n + \frac{5}{2}; \quad \mathit{for}\ n \geq -2$$

The complete response is simply the sum of zero-state and zero-input responses (considering only $n \geq 0$):

$$\begin{aligned}
y_c(n) &= y_{zs} + y_{zi} \\
&= \frac{3}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^n + \frac{5}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^n \\
&= 4 - \left(\frac{1}{3}\right)^n \\
&\quad \mathit{for}\ n \geq 0
\end{aligned}$$

3. (a) The z-transform of the input sequence $x(n) = u(n)$ is:

$$X(z) = \frac{z}{z-1} \quad \textit{ROC: } |z| > 1$$

Compute the z-transform of the sequence $y(n) = \left(\frac{1}{2}\right)^n u(n+1) = 2\left(\frac{1}{2}\right)^{n+1} u(n+1)$.

$$Y(z) = 2z \times \frac{z}{z-1/2} = \frac{2z^2}{z-1/2} \quad \textit{ROC: } |z| > \frac{1}{2} \textit{ excluding } \infty$$

The transfer function of the system is:

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{2z(z-1)}{z-1/2} \\ &= \frac{2z(z-1/2)-z}{z-1/2} \\ &= 2z - \frac{z}{z-1/2} \end{aligned}$$

$$\textit{ROC: } |z| > \frac{1}{2} \textit{ excluding } \infty$$

Use inverse z-transform to obtain the impulse response.

$$h(n) = 2\delta(n+1) - \left(\frac{1}{2}\right)^n u(n)$$

(b) The system has two poles (1/2, ∞) and two zeros (0, 1). They are plotted in Figure 1.

(c) The system is stable because the ROC includes the unit circle and the impulse response $h(n)$ is absolutely summable.

(d) The system is not causal. Again, if you look at the ROC, it is not the entire region outside the circle ($|z| = 1/2$). On the other hand the impulse response has a non-zero value at $n = -1$, which also indicates that the system is not causal.

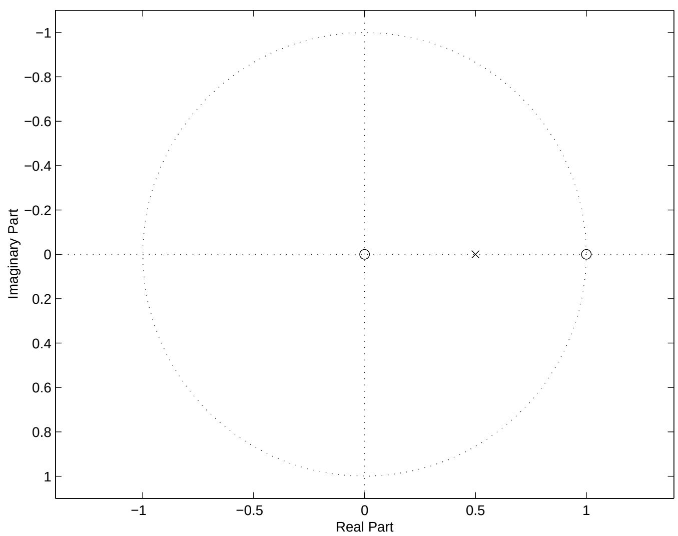

Figure 1: The zero and pole plot of transfer function $H(z)$. Circles represent zeros, and cross indicates pole. The pole ∞ is not shown in the plot

4. (a) By looking at the four sequences, we find that

$$\begin{aligned} 2x_3(n) - x_1(n-1) &= \{2, 0, \boxed{0}\} + \{0, \boxed{1}, -1\} \\ &= \{2, 0, \boxed{-1}, 1\} = x_2(n) \end{aligned}$$

Since the system is time-invariant, then we know that it output to $x_1(n-1)$ is $y(n-1)$. Now, if $2y_3(n) - y_1(n-1) = y_2(n)$, the system can be linear otherwise, the system is nonlinear.

$$\begin{aligned} 2y_3(n) - y_1(n-1) &= \{2, \boxed{-2}\} - \{\boxed{0}, 1, -3, 2\} \\ &= \{2, \boxed{-2}, 1, -3, 2\} \end{aligned}$$

and

$$y_2(n) = \{2, \boxed{-4}, 1, -3, 2\}$$

Then,

$$2y_3(n) - y_1(n-1) \neq y_2(n)$$

and the system is <u>nonlinear</u>.

(b) We can re-write the relation in part (a) as

$$x_1(n-1) = 2x_3(n) - x_2(n)$$

Since the system is linear then the output to the linear combination of $x_2(n)$ and $x_3(n)$ is the same linear combination of $y_2(n)$ and $y_3(n)$. i.e.,

$$\begin{aligned} \mathcal{S}[2x_3(n) - x_2(n)] &= 2y_3(n) - y_2(n) \\ &= \{2, \boxed{-2}\} - \{2, \boxed{-4}, 1, -3, 2\} \\ &= \{\boxed{2}, 1, -3, 2\} = \mathcal{S}[x_1(n-1)] \end{aligned}$$

We can also show that

$$y_1(n-1) = \{\boxed{0}, 1, -3, 2\}$$

Hence

$$\mathcal{S}[x_1(n-1)] \neq y_1(n-1)$$

and the system is <u>time-variant</u>.

(c) The system is linear and time-invariant. In this case, the output of the system to any input can be found be convolving the input with the impulse response sequence. By inspecting $x_1(n)$ and $y_1(n)$, we can show that

$$h(n) = \delta(n-1) - 2\delta(n-2)$$

This result can be obtained in many different ways

i. The output of an LTI system in terms of the input sequence and the impulse response sequence is given by the convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Now if we look at the first sequence, we find that $x(n)$ is nonzero within the interval $-1 \leq n \leq 0$ and $y(n)$ is nonzero within the interval $0 \leq n \leq 2$. We conclude that $h(n)$ is nonzero within the interval $1 \leq n \leq 2$. Using this observation and the convolution sum we get

$$\begin{aligned}
y(0) &= \sum_{k=-\infty}^{\infty} x(k)h(-k) = x(-1)h(1) \\
&= \boxed{h(1) = 1} \\
y(1) &= \sum_{k=-\infty}^{\infty} x(k)h(1-k) = x(0)h(1) + x(-1)h(2) \\
&= \boxed{-h(1) + h(2) = -3} \\
y(1) &= \sum_{k=-\infty}^{\infty} x(k)h(2-k) = x(0)h(2) \\
&= \boxed{-h(2) = 2}
\end{aligned}$$

By solving these equations we find that $h(1) = 1$ and $h(2) = -2$. hence,

$$h(n) = \delta(n-1) - 2\delta(n-2)$$

ii. by using the z-transform as follows

- Find the z-transform of the first input output pair

$$\begin{aligned}
X_1(z) &= z - 1 \\
Y_1(z) &= 1 - 3z^{-1} + 2z^{-2}
\end{aligned}$$

- Find the transfer function $H(z) = \dfrac{Y(z)}{X(z)}$

$$H(z) = \frac{1 - 3z^{-1} + 2z^{-2}}{z - 1} = \frac{(z-1)(z-2)}{z^2(z-1)} = \frac{1}{z} - \frac{2}{z^2}$$

- use the inverse z-transform to find $h(n)$

$$h(n) = \delta(n-1) - 2\delta(n-2)$$

We can get the same result by using any other input-output pair.

iii. Since we know that the system is LTI, we can use any linear combination on the inputs that gives $\delta(n)$ and find the corresponding output. One of these combinations will be

$$\delta(n) = \frac{1}{2}(x_4(n) - x_2(n) - x_1(n))$$

Therefore,

$$\begin{aligned} h(n) &= \frac{1}{2}(y_4(n) - y_2(n) - y_1(n)) \\ &= \delta(n-1) - 2\delta(n-2) \end{aligned}$$