Name: Solution

UID:

UCLA
Dept. of Electrical Engineering
EE113: Digital Signal Processing
Midterm Exam, Fall 2011

This exam consists of five problems. No calculators, phones, or any other electronic devices are allowed. One 8.5 x 11 sheet (double sided) is allowed into the exam. Please justify your answers clearly; a correct answer with no justification will not receive full credit. Please write your name on each page. Good Luck!

1. (24 points) The various parts of this question are independent of one another.

   (A) For a particular system, we observe that when the input is:
   $x(n) = e^{jn/8}u(n)$
   the output is:
   $y(n) = 2e^{jn/8}u(n)$
   Can the system be LTI? Explain your reason.

Solution: input: $x(n) = e^{j\frac{n}{8}} u(n)$

output: $y(n) = 2e^{jn/8} u(n)$

thus, $y(n) = 2x(n)$, which is an amplifier

1) Linearity:

Let $x(n) = \alpha \cdot x_1(n) + \beta x_2(n)$ and $y_1(n) = 2x_1(n)$, $y_2(n) = 2x_2(n)$

$\therefore y(n) = 2x(n) = 2[\alpha x_1(n) + \beta x_2(n)]$

$= \alpha \cdot 2x_1(n) + \beta \cdot 2x_2(n)$

$= \alpha y_1(n) + \beta \cdot y_2(n)$

$\Rightarrow$ This system is <u>Linear</u>

2) Time-invariant:

Let $x_1(n) = x(n-n_0)$

$\Rightarrow y_1(n) = 2x_1(n) = 2x(n-n_0) = y(n-n_0)$

$\Rightarrow$ This system is Time-invariant

from 1) and 2) The system is LTI

(B) The continuous-time signal:

$x(t) = cos(2000\pi t)$

is sampled with a certain sampling frequency to produce the discrete-time signal:

$x(n) = cos(\pi n/3)$.

What sampling frequency was used? Is the answer unique?

Solution: 1) Let f_s denotes the sampling frequency. we have

$$x(n) = x(t)\big|_{t=n/f_s} = \cos(2000\pi t)\big|_{t=n/f_s} = \cos(2000\pi n/f_s)$$

$\because$ $x(n) = \cos(\pi n/3)$ and it is periodic. we get:

$$\frac{2000\pi n}{f_s} = \frac{\pi n}{3} + 2k\pi \quad , \quad k \text{ is an integer} \qquad (*)$$

Let $k=0$, $\Rightarrow$ $f_s = 6000\,Hz$

2) From $(*)$, we conclude the answer is <u>not unique</u>.

(C) Let the impulse response of an LTI system be $h(n)$ where $(.5)^n < h(n) < (5/6)^n$ for $n \geq 0$, and $h(n) = 0$ for $n < 0$.

Is this system BIBO? Is it Causal? Explain.

Solution:

- This system is <u>causal</u>, since $h(n)=0$ for $n<0$

- From the given condition

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n < \sum_{n=0}^{\infty} |h(n)| < \sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n$$

$$\Rightarrow \quad \frac{1}{1-\frac{1}{2}} < \sum_{n=0}^{\infty} |h(n)| < \frac{1}{1-\frac{5}{6}}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} |h(n)| \text{ is finite}$$

$\Rightarrow$ System is BIBO stable.

2. (20 points)

Consider the LTI system with impulse response: $h(n) = a^{-n}u(-n)$

where $0 < a < 1$.

Find the step response of this system without using the Z-transform.

Solution:

We desire the step response to a system whose impulse response in

$$h(n) = a^{-n} u(-n), \qquad 0<a<1$$

The convolution sum:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

The step response results when the input is the unit step:

$$x(n) = u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n<0 \end{cases}$$

$$\Rightarrow \quad y(n) = \sum_{k=-\infty}^{\infty} a^{-k} u(-k)\, u(n-k)$$

For $n \leq 0$;

$$y(n) = \sum_{k=-\infty}^{n} a^{-k}$$

$$= \sum_{k=-n}^{\infty} a^{k}$$

$$= \frac{a^{-n}}{1-a}$$

For $n > 0$;

$$y(n) = \sum_{k=-\infty}^{0} a^{-k}$$

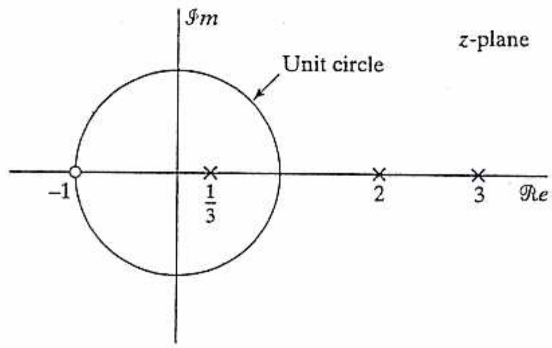
$$= \sum_{k=0}^{\infty} a^{k}$$

$$= \frac{1}{1-a}$$

$$\Rightarrow \quad y(n) = \begin{cases} \dfrac{a^{-n}}{1-a}, & n \leq 0 \\ \dfrac{1}{1-a} & n>0 \end{cases}$$

3. **(16 points)**

Below is the pole-zero plot for the transfer function of an an LTI system.



8' (a) How many possible two-sided sequences have the pole-zero plot shown in the figure?

8' (b) If the system is causal, can it be also stable? Explain.

Solution: (a). There are two possible two-sided sequences, which correspend to the following two ROCs, respectively:

① $\text{ROC} = \{ z: \frac{1}{3} < |z| < 2 \}$

② $\text{ROC} = \{ z: 2 < |z| < 3 \}$

(b). No. If the system is causal, then the ROC has to be:

$$\text{ROC} = \{ z: |z| > 3 \}$$

which does not include the unit circle. Hence, it won't be stable.

4. **(20 points)**

If the system response of a causal LTI system has the following form:

$H(z) = \frac{1}{(1+.5z^{-1})(1-2z^{-1})}$

6' (a) What is the underlying difference equation that relates the input to the output?

7' (b) Plot the pole-zero diagram for this system. Find the ROC.

7' (c) Find the impulse response, $h(n)$.

Solution: (a) Since $H(z) = \frac{Y(z)}{X(z)}$, we have

$$\frac{Y(z)}{X(z)} = \frac{1}{(1+0.5z^{-1})(1-2z^{-1})}$$

$$(1+0.5z^{-1})(1-2z^{-1})\,Y(z) = X(z)$$

$$Y(z) - 1.5z^{-1}Y(z) - z^{-2}Y(z) = X(z).$$

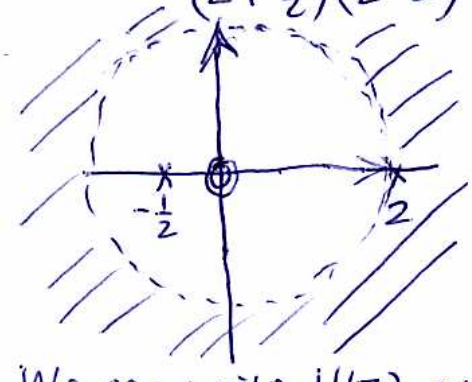
The corresponding difference equation would be

$$\boxed{y(n) = 1.5y(n-1) + y(n-2) + x(n).}$$

(b). $$H(z) = \frac{z^2}{(z+\frac{1}{2})(z-2)}$$

zeros: $z=0$, $z=0$.

poles: $z=-\frac{1}{2}$, $z=2$.



Since the system is causal, the ROC has to be:

$$ROC = \{z : |z| > 2\}$$

(c). We can write $H(z)$ as: $H(z) = z \cdot \underbrace{\left[\frac{z}{(z+\frac{1}{2})(z-2)}\right]}_{H_0(z)}$

~~We can first find $h_0(n) = \mathcal{Z}^{-1}\{H_0(z)\}$, then $h(n) = h_0(n+1)$~~

$$H_0(z) = \frac{z}{(z+\frac{1}{2})(z-2)} = \frac{A}{z+\frac{1}{2}} + \frac{B}{z-2} \Rightarrow \begin{cases} A = H_0(z)(z+\frac{1}{2})\big|_{z=-\frac{1}{2}} = \frac{1}{5} \\ B = H_0(z)(z-2)\big|_{z=2} = \frac{4}{5} \end{cases}$$

Then $$H_0(z) = \frac{\frac{1}{5}}{z+\frac{1}{2}} + \frac{\frac{4}{5}}{z-2}$$

$$H(z) = z \cdot H_0(z) = \frac{1}{5} \cdot \frac{z}{z+\frac{1}{2}} + \frac{4}{5} \cdot \frac{z}{z-2}$$

The ROC of the system is $|z| > 2$, as given in part (b). Therefore,

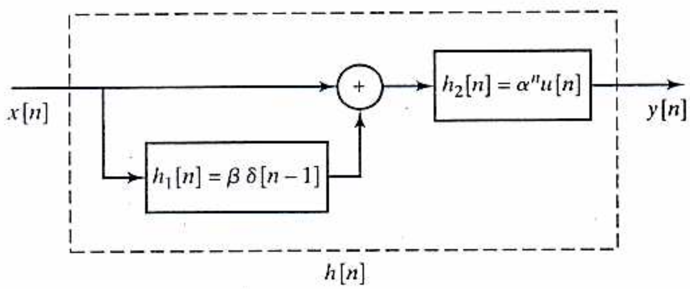
$$h(n) = \mathcal{Z}^{-1}\{H(z)\} = \left[\frac{1}{5} \cdot \left(-\frac{1}{2}\right)^n + \frac{4}{5} 2^n\right] u(n).$$

5. **(20 points)**

Consider the system in the figure below.

(a) Find the impulse response of the overall system without using the $Z-$ *transform*.

(b) Find the transfer function, $H(z)$, of the overall system.



Solution : (a).

$$y(n) = h_2(n) * [x(n) + h_1(n) * x(n)]$$

$$= \underbrace{h_2(n) * [\delta(n) + h_1(n)]}_{h(n)} * x(n)$$

$$h(n) = h_2(n) * [\delta(n) + h_1(n)]$$

$$= \alpha^n u(n) * [\delta(n) + \beta\delta(n-1)]$$

$$= \alpha^n u(n) + \beta \cdot \alpha^{n-1} u(n-1)$$

$$= \delta(n) + (\alpha+\beta)\,\alpha^{n-1} u(n-1)$$

(b).

$$H(z) = \mathcal{Z}\{h(n)\} = 1 + (\alpha+\beta)\, z^{-1} \cdot \frac{z}{z-\alpha}$$

$$= 1 + \frac{\alpha+\beta}{z-\alpha}$$

$$= \frac{z+\beta}{z-\alpha} \qquad (|z| > \alpha)$$