EE113: Digital Signal Processing Fall 2012
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MIDTERM EXAMINATION SOLUTION

This exam consists of five problems. Please justify your answers clearly; a correct answer with no justification will not receive full credit. Please write your name clearly on each page. Good Luck!

1. For the pair of sequences in Fig. 1, use discrete convolution to find the response to the input $x(n)$ of the linear time-invariant system with impulse response $h(n)$. Do not use the z-transform.

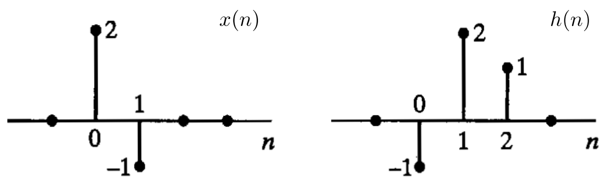

Figure 1: Figure for Problem 1.

Solution: We can interprete $x(n)$ as

$$x(n) = 2\delta(n) - \delta(n-1)$$

Then $y(n) = h(n) * x(n)$ would be

$$\begin{aligned} y(n) &= h(n) * x(n) \\ &= h(n) * [2\delta(n) - \delta(n-1)] \\ &= 2h(n) - h(n-1) \end{aligned}$$

We know $h(n) = -\delta(n) + 2\delta(n-1) + \delta(n-2)$. Then,

$$\begin{aligned} y(n) &= 2[-\delta(n) + 2\delta(n-1) + \delta(n-2)] - [-\delta(n-1) + 2\delta(n-2) + \delta(n-3)] \\ &= -2\delta(n) + 4\delta(n-1) + 2\delta(n-2) + \delta(n-1) - 2\delta(n-2) - \delta(n-3) \\ &= -2\delta(n) + 5\delta(n-1) - \delta(n-3) \end{aligned}$$

2. The signals $x(n)$ and $y(n)$ shown in Fig. 2 are the input and corresponding output for an LTI system.

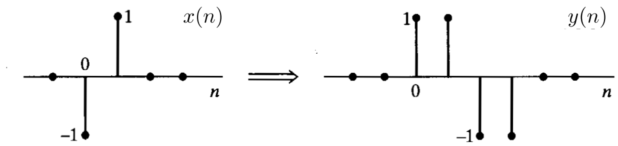

Figure 2: The input and corresponding output sequence for Problem 2.

(a) Find the impulse response $h(n)$ for this LTI system by using the z-transform. Is $h(n)$ FIR or IIR?

(b) Find the response of the system to the sequence $w(n)$ in Fig. 3.

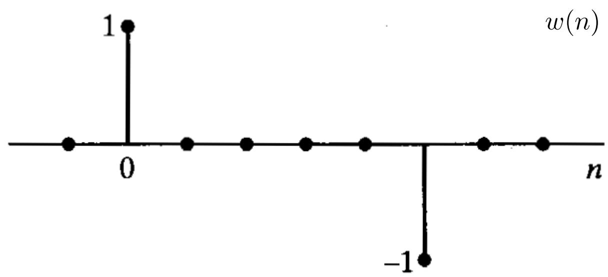

Figure 3: Input sequence $w(n)$ for Problem 2.

Solution:

(a) We can model $x(n)$ and $y(n)$ respectively as

$$x(n) = -\delta(n) + \delta(n-1)$$
$$y(n) = \delta(n) + \delta(n-1) - \delta(n-2) - \delta(n-3)$$

We can find the z-transform of each signal by using the time-shift property to be:

$$X(z) = -1 + z^{-1}$$
$$Y(z) = 1 + z^{-1} - z^{-2} - z^{-3}$$

We can now find the transfer function $H(z)$ as:

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{1 + z^{-1} - z^{-2} - z^{-3}}{-1 + z^{-1}} \\ &= \frac{1}{-1+z^{-1}} + \frac{z^{-1}}{-1+z^{-1}} - \frac{z^{-2}}{-1+z^{-1}} - \frac{z^{-3}}{-1+z^{-1}} \end{aligned}$$

Using the causal transform pair

$$a^n u(n) \Longleftarrow \frac{1}{1-z^{-1}}$$

we obtain the inverse z-transform:

$$\begin{aligned} h(n) &= -u(n) - u(n-1) + u(n-2) + u(n-3) \\ &= -\delta(n) - 2\delta(n-1) - \delta(n-2) \end{aligned} \tag{1}$$

(b) To find the convolution of $w(n)$ and $h(n)$, we notice that $w(n)$ can be modeled as the summation of two impulse sequences:

$$w(n) = \delta(n) - \delta(n-5)$$

Now, using $h(n) \star \delta(n-n_0) = h(n-n_0)$, we obtain (using (1):

$$\begin{aligned} y(n) &= h(n) - h(n-5) \\ &= -\delta(n) - 2\delta(n-1) - \delta(n-2) + \delta(n-5) + 2\delta(n-6) + \delta(n-7) \end{aligned}$$

3. The signal $y(n)$ is the output of an LTI system with impulse response $h(n)$ for a given input $x(n)$. Throughout the problem, assume that $y(n)$ is stable and has a z-transform $Y(z)$ with the pole-zero diagram shown on the left of Fig. 4. The signal $x(n)$ is stable and has the pole-zero diagram shown on the right of Fig. 4.

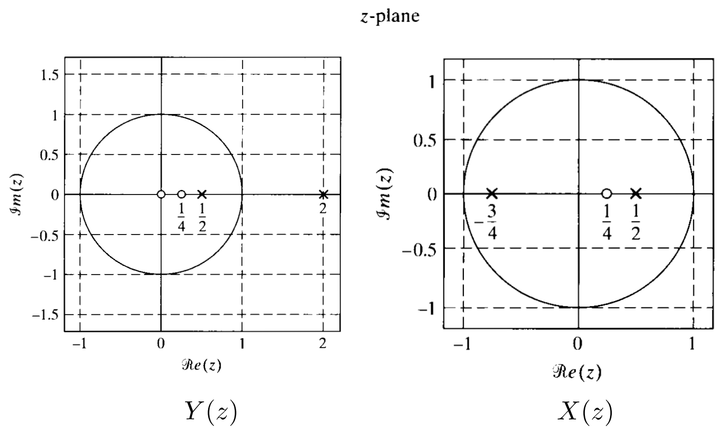

Figure 4: Pole-zero plot for $Y(z)$ (left) and $X(z)$ (right) for Problem 3.

(a) What is the ROC of $Y(z)$?
(b) Is $y(n)$ left-sided, right-sided, or two-sided?
(c) What is the ROC of $X(z)$?
(d) Is $x(n)$ a causal sequence? That is, does $x(n) = 0$ for $n < 0$?
(e) What is $x(0)$?
(f) Draw the pole-zero plot of $H(z)$, and specify its ROC.
(g) Is $h(n)$ anti-causal? That is, does $h(n) = 0$ for $n > 0$?

Solution:

(a) Since $y(n)$ is stable, then the ROC of $Y(z)$ should contain the unit circle. Therefore, the ROC is

$$\text{ROC} = \{z : \frac{1}{2} < |z| < 2\}$$

(b) Since the ROC of $Y(z)$ is a ring, it should be two sided.
(c) Since $x(n)$ is stable, the ROC of $X(z)$ should contain unit circle. Therefore, the ROC is

$$\text{ROC} = \{z : |z| > \frac{3}{4}\}$$

(d) Yes, because the ROC of $X(z)$ is outside a circle.
(e) Since $x(n)$ is causal, we can apply initial value theorem and get $x(0)$ as

$$x(0) = \lim_{z\to\infty} X(z)$$

Note that there are two finite poles and one finite zero for $X(z)$. Thus, there must be a zero at ∞ so that

$$x(n) = \lim_{z\to\infty} X(z) = 0$$

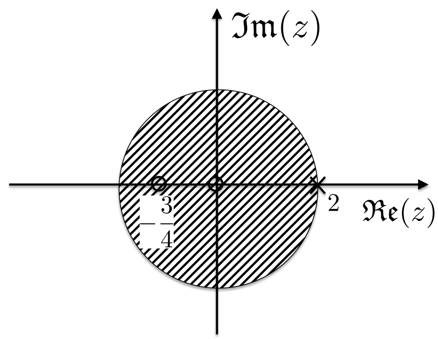

Figure 5: The zero-pole plot and ROC of $H(z)$.

(f) The ROC is

$$\text{ROC} = \{z : \ |z| < 2\}$$

The zero-pole plot and the ROC are shown in Fig. 5.

(g) Yes. $h(n)$ is anticausal because the ROC of $H(z)$ is inside a circle.

4. When the input to a linear time-invariant system is

$$x(n) = \left(\frac{1}{2}\right)^n u(n) + 2^n u(-n-1)$$

the output is

$$y(n) = 6\left(\frac{1}{2}\right)^n u(n) - 6\left(\frac{3}{4}\right)^n u(n)$$

(a) Find the system function $H(z)$ of the system. Plot the poles and zeros of $H(z)$, and indicate the region of convergence.

(b) Find the impulse response $h(n)$ of the system for all values of n.

(c) Write the difference equation that characterizes the system.

(d) Is the system stable? Is it causal?

Solution:

(a) Taking the $z-$transofrm of $x(n)$ and $y(n)$, respectively, we obtain:

$$\begin{aligned} X(z) &= \frac{z}{z-\frac{1}{2}} - \frac{z}{z-2} \\ &= \frac{-\frac{3}{2}z}{(z-\frac{1}{2})(z-2)}, \qquad \frac{1}{2} < |z| < 2 \\ Y(z) &= 6\frac{z}{z-\frac{1}{2}} - 6\frac{z}{z-\frac{3}{4}} \\ &= \frac{-\frac{3}{2}z}{(z-\frac{1}{2})(z-\frac{3}{4})}, \qquad |z| > \frac{3}{4} \end{aligned}$$

Then, the system function $H(z)$ can be expressed as

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{\frac{-\frac{3}{2}z}{(z-\frac{1}{2})(z-\frac{3}{4})}}{\frac{-\frac{3}{2}z}{(z-\frac{1}{2})(z-2)}} \\ &= \frac{z-2}{z-\frac{3}{4}}, \qquad |z| > \frac{3}{4} \end{aligned}$$

The zero-pole plot and the ROC of $H(z)$ are shown in Fig. 6.

(b) First, we express $H(z)$ as

$$\begin{aligned} H(z) &= \frac{z-\frac{3}{4}-\frac{5}{4}}{z-\frac{3}{4}} \\ &= 1 - \frac{\frac{5}{4}}{z-\frac{3}{4}} \\ &= 1 - z^{-1}\frac{\frac{5}{4}z}{z-\frac{3}{4}} \end{aligned}$$

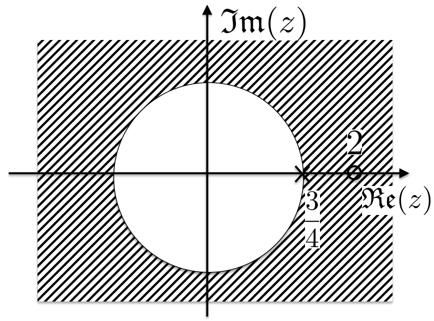

Figure 6: The zero-pole plot and the ROC of $H(z)$.

with ROC being $\{z: \ |z| > 3/4\}$. Then, taking the inverse $z-$transform, we obtain

$$h(n) = \delta(n) - \frac{5}{4}\left(\frac{3}{4}\right)^{n-1} u(n-1)$$

(c) Note that

$$\frac{Y(z)}{X(z)} = H(z) = \frac{z-2}{z-\frac{3}{4}} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}$$

Therefore, we have

$$\left(1 - \frac{3}{4}z^{-1}\right) Y(z) = (1-2z^{-1})X(z)$$

Taking the inverse $z-$transform of both sides of the above equation, we obtain

$$y(n) - \frac{3}{4}y(n-1) = x(n) - 2x(n-1)$$

which is equivalent to

$$y(n) = \frac{3}{4}y(n-1) + x(n) - 2x(n-1)$$

(d) The system is stable, because its ROC contains the unit circle. And the system is causal, because its ROC is outside a circle ($|z| > \frac{3}{4}$).

5. The grandparents of John Doe decided to open a savings account for the child. They opened the account when he was born with a deposit of \$2000. On his birthday every year thereafter, they would deposit another \$2000. Assume that the account pays 10% interest compounded annually. The difference equation that describes the evolution of his saving account can be written as:

$$y(n) = 1.1y(n-1) + x(n)$$

(a) Explain what are $x(n)$ and $y(n)$ in the equation?

(b) Draw a block diagram of this difference equation.

(c) Express the equation as a finite summation (closed form).

(d) Is this a linear system? Is it time invariant?

Solution:

(a) $x(n)$ is a sequence of 2000 starting at time 0 and ending at time n.
$y(n)$ is the amount of money at time n in the savings account.

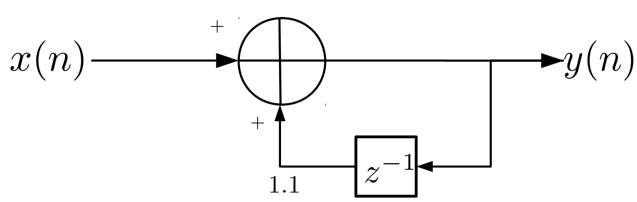

Figure 7: Block diagram of the system

(b)

(c) We can find the summation by making the following table: and by inspection, we can

n	$x(n)$	$y(n-1)$	$y(n)$
-1	$x(-1)$	0	0
0	$x(0)$	0	$x(0)$
1	$x(1)$	$x(0)$	$1.1x(0) + x(1)$
2	$x(2)$	$1.1x(0) + x(1)$	$(1.1)^2x(0) + 1.1x(1) + x(2)$
$\vdots$			

deduce that the output $y(n)$ can be found as

$$y(n) = \sum_{k=0}^{n} (1.1)^{n-k} x(k) \qquad (2)$$

When $x(k) = 2000$ for $0 \leq k \leq n$, we have

$$\begin{aligned}
y(n) &= 2000 \sum_{k=0}^{n} (1.1)^{n-k} \\
&= 2000(1.1)^n \sum_{k=0}^{n} \left(\frac{1}{1.1}\right)^k \\
&= 2000(1.1)^n \frac{1 - \left(\frac{1}{1.1}\right)^{n+1}}{1 - \frac{1}{1.1}} \\
&= 2000 \frac{(1.1)^n - \frac{1}{1.1}}{1 - \frac{1}{1.1}} \\
&= 2000 \frac{(1.1)^{n+1} - 1}{1.1 - 1} \\
&= 2000 \frac{(1.1)^{n+1} - 1}{0.1} \\
&= 20000(1.1)^{n+1} - 20000
\end{aligned}$$

(d) The system is linear and time-invariant since it is described by a relaxed ($y(-1) = 0$) linear difference equation.