EE 110
CIRCUIT ANALYSIS II
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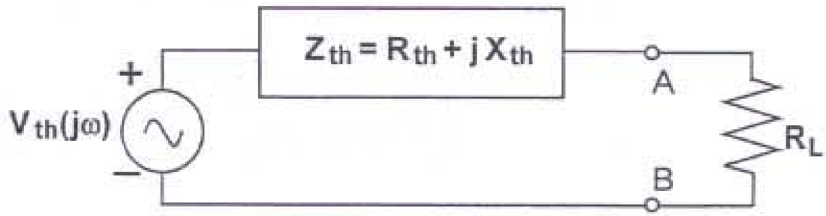
MID-TERM EXAMINATION

Do all work in this examination packet. There are four questions. Each counts 25 points. Good luck!

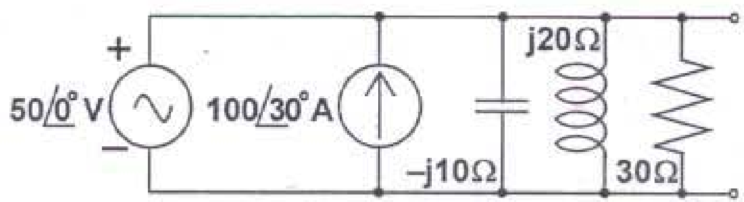
1A. (10 points) The circuit to the left of nodes A and B has a Thevenin impedance Z_{th} for which $R_{th} > 0$ and $X_{th} \neq 0$. The resistor R_L is required to have a value that will maximize the power that will be delivered to it.



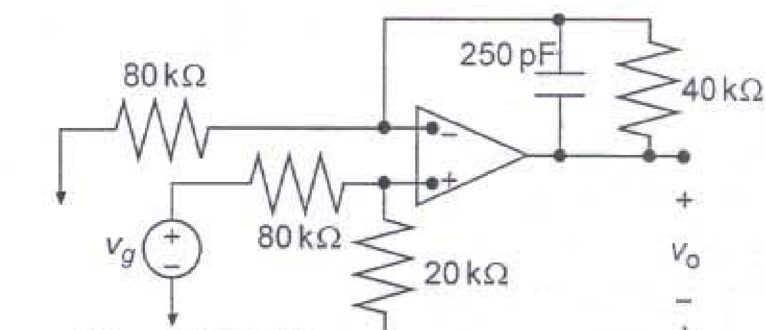
a.) Derive a formula for the required value of R_L as a function of the R_{th} and X_{th} components of the Thevenin impedance Z_{th}.

b.) Let $R_{th} = 12\,\Omega$ and $X_{th} = 5\,\Omega$, and use your derived formula to find this optimum value of R_L.

1B. (5 points) Find the Thevenin equivalent circuit for this circuit:

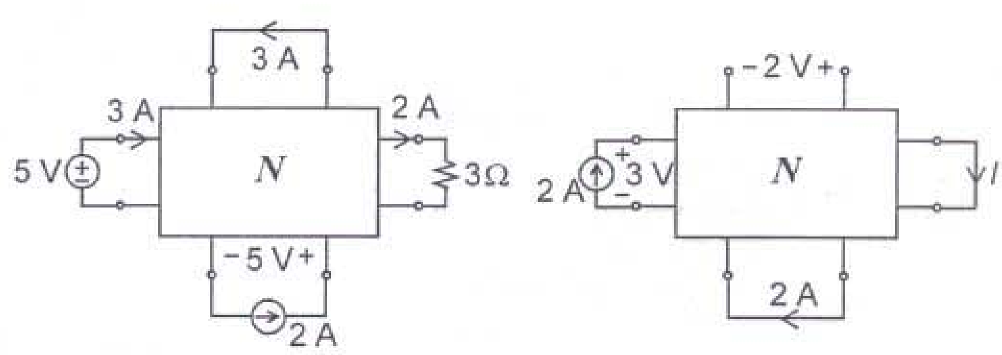


1C. (5 points) The op amp in this circuit is ideal. Find the steady-state expression for $v_o(t)$.



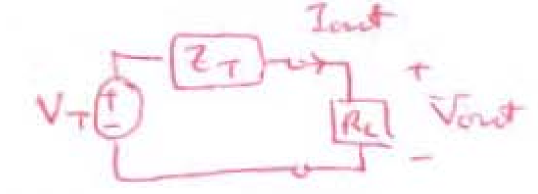
$v_g = 25\cos 50{,}000t$ V

1D. (5 points) When an eight-terminal resistive network N is subjected to various external connections, the results are as indicated in the figure below. Determine the current I.



1A. a)

$$V_{out} = \frac{R_L}{R_T + jX_T + R_L} V_T$$



Power delivered $= P = \frac{1}{2} \mathrm{Re}\left(V_{out} * I_{out}^{*} \right)$

$$= \frac{1}{2} \mathrm{Re}\left(\frac{R_L}{R_T + jX_T + R_L} V_T \frac{V_T^{*}}{(R_T + jX_T + R_L)^{*}} \right)$$

$$= \frac{1}{2} |V_T|^2 \frac{R_L}{|R_T + jX_T + R_L|^2}$$ ← To maximize this, set deriv w.r.t. R_L to zero.

$$\frac{\partial P}{\partial R_L} = \frac{|\ |^2 \times 1 - R_L 2 (R_T + R_L)}{\left[(R_T + R_L)^2 + X_T^2 \right]^2}$$

$|R_T + jX_T + R_L|^2 = (R_T + R_L)^2 + X_T^2$

$$= 0 \Leftrightarrow (R_T + R_L)^2 + X_T^2 = 2R_T R_L + 2R_L^2$$

$$\Leftrightarrow R_T^2 + \cancel{2R_T R_L} + R_L^2 + X_T^2 = \cancel{2R_T R_L} + 2R_L^2$$

$$\Leftrightarrow R_T^2 + X_T^2 = R_L^2 \Leftrightarrow R_L = \sqrt{R_T^2 + X_T^2}$$

$$\Leftrightarrow \boxed{R_L = |R_T + jX_T|}$$

$$\Leftrightarrow \boxed{R_L = |Z_T|}$$

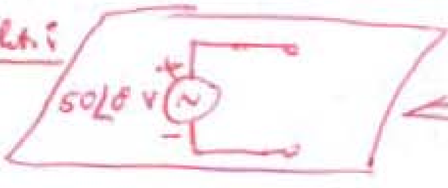
b) $R_L = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13.$

$$\boxed{R_L = 13\ \Omega}$$

1B. To get Thev. Impedance: 1) Set indep sources to zero

2) Get open-ckt voltage: $V_{oc} = 50\angle 0^\circ$

$\Rightarrow$ Thev. Eq. Ckt:



← That's IT!!

1c. $V_g = 25\angle 0^\circ \text{ V} \Rightarrow V_+ = \frac{20}{100}V_g = 5\angle 0^\circ$;

$\therefore$ $V_- = V_+ = 5\angle 0^\circ$ V.

So: $\frac{5}{80{,}000} + \frac{5 - V_o}{Z_p} = 0$

$\frac{1}{j\underset{5\times10^4}{\omega} \underset{250\times10^{-12}}{C}} = -j80{,}000$

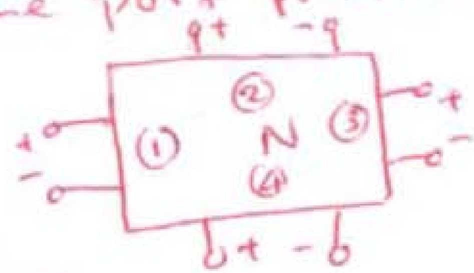
$Z_p = -j80{,}000 \,\|\, 40{,}000$

$= 32{,}000 - j16{,}000\ \Omega$

$V_o = \frac{5Z_p}{80{,}000} + 5 = 7 - j1 = 7.07\angle -8.13^\circ$ V

$\Rightarrow \boxed{v_o(t) = 7.07\cos(50{,}000t - 8.13^\circ) \text{ V}}$

1D. Use same port polarities for both circuits:



Use Thev. Corollary B

$\boxed{\sum_{4\text{ports}} v_L i_R = \sum_{4\text{ports}} v_R i_L}$

From given diagrams:

	ports ①	②	③	④		①	②	③	④
v_L	5	0	6	−5	v_R	3	−2	0	0
i_L	3	3	−2	−2	i_R	2	0	−I	2

$\sum v_L i_R = \cancel{5\times2} + 0\times0 + 6\times(-I) + \cancel{(-5)\times2} = -6I$

$\sum v_R i_L = 3\times3 + (-2)\times3 + 0\times(-2) + 0\times(-2) = 9 - 6 = 3$

So: $-6I = 3 \Rightarrow \boxed{I = -\frac{1}{2} \text{ A}}$

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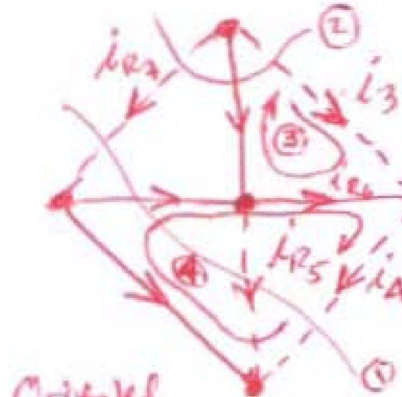
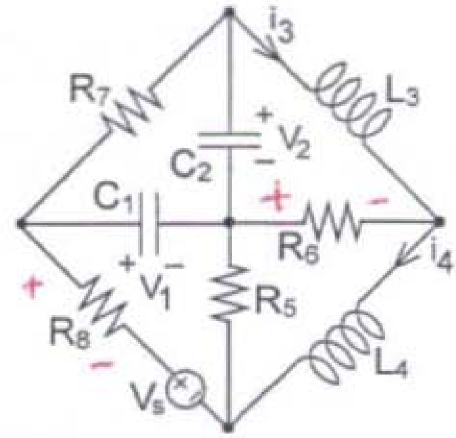
2. (25 points) Write state equations for this circuit.





Oriented Graph of circuit having a proper tree.

$\boxed{\text{state variables are } v_1, v_2, i_3, i_4}$

We need to express some resistor currents and/or voltages in terms of statevariables.

Fundamental cut-set equations: (for the selected tree)

① $C_1 \frac{dv_1}{dt} = i_4 + i_{R5} + i_{R7}$

② $C_2 \frac{dv_2}{dt} = -i_3 - i_{R7}$

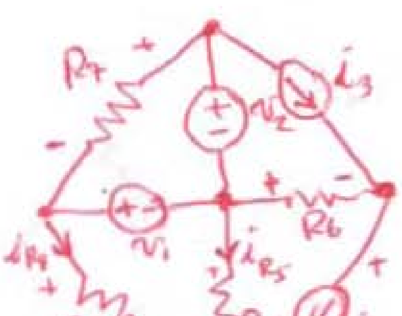
Fundamental loop eqns (for tree):

③ $L_3 \frac{di_3}{dt} = V_2 + v_{R6}$

④ $L_4 \frac{di_4}{dt} = -V_1 - v_{R6} + v_{R8} + v_s$

We need (i_{R5}, i_{R7}, v_{R6} & v_{R8}).

This "graph" containing "sources" & resistors can help:



Easy ones: $i_{R7} = \frac{1}{R_7}(v_2 - v_1)$; $v_{R6} = R_6(i_4 - i_3)$

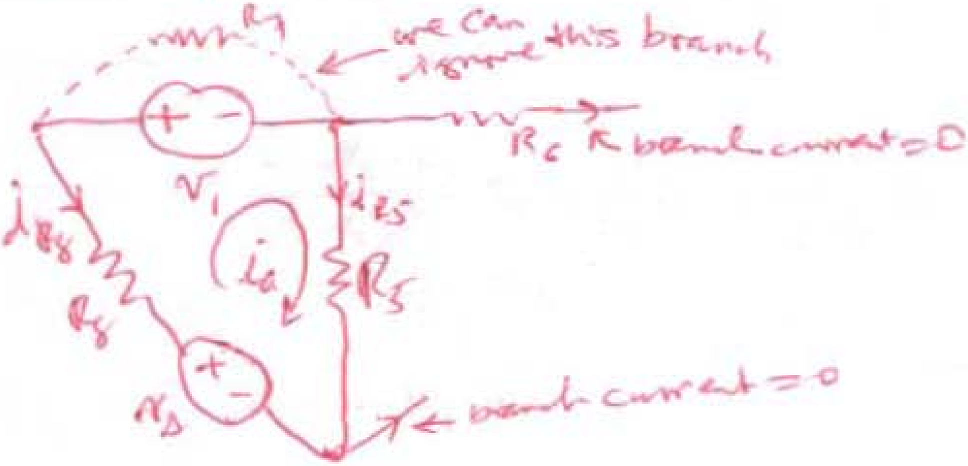
The other two: We can write two simultaneous eqns. in order to solve for i_{R5} & i_{R8}. Here's another way: (Use superposition) First, just turn on source i_4. Then, we have this circuit

By current division:

$i_{R5} = -\left(\frac{R_8}{R_5+R_8}\right) i_4$

$i_{R8} = -\left(\frac{R_5}{R_5+R_8}\right) i_4$

Similarly, if we just turn on v_1 and v_s, we have



$$\text{Loop current } i_a = \frac{v_s - v_1}{R_5 + R_8} = -i_{R_8} = i_{R_5}$$

The other two sources (v_2 & i_3), if turned on exclusively (by themselves), will not cause currents i_{R_5} & i_{R_8} to flow.

Thus, the sought after resistor currents are:

$$i_{R_5} = -\left(\frac{R_8}{R_5+R_8}\right) i_4 + \left(\frac{1}{R_5+R_8}\right)(v_s - v_1)$$

$$\& \quad i_{R_8} = -\left(\frac{R_5}{R_5+R_8}\right) i_4 - \left(\frac{1}{R_5+R_8}\right)(v_s - v_1)$$

So:

$$\begin{cases} C_1 \frac{dv_1}{dt} = i_4 - \left(\frac{R_8}{R_5+R_8}\right) i_4 + \left(\frac{1}{R_5+R_8}\right) v_s - \left(\frac{1}{R_5+R_8}\right) v_1 + \frac{1}{R_7} v_2 - \frac{1}{R_7} v_1 \\ C_2 \frac{dv_2}{dt} = -i_3 - \frac{1}{R_7} v_2 + \frac{1}{R_7} v_1 \\ L_3 \frac{di_3}{dt} = v_2 + R_6 i_4 - R_6 i_3 \\ L_4 \frac{di_4}{dt} = -v_1 + R_6 i_3 - R_6 i_4 - \left(\frac{R_5 R_8}{R_5+R_8}\right) i_4 + \left(\frac{R_8}{R_5+R_8}\right) v_1 - \left(\frac{R_8}{R_5+R_8}\right) v_s + v_s \end{cases}$$

This can be "cleaned up" a bit by combining some terms:

$$
\begin{cases}
C_1 \dfrac{dv_1}{dt} = \dfrac{R_5}{R_5+R_8} i_4 - \dfrac{R_5+R_8+R_7}{(R_5+R_8)R_7} v_1 + \dfrac{1}{R_7} v_2 + \dfrac{1}{R_5+R_8} v_s \\
C_2 \dfrac{dv_2}{dt} = -i_3 + \dfrac{1}{R_7} v_1 - \dfrac{1}{R_7} v_2 \\
L_3 \dfrac{di_3}{dt} = v_2 - R_6 i_3 + R_6 i_4 \\
L_4 \dfrac{di_4}{dt} = -\dfrac{R_5}{R_5+R_8} v_1 + R_6 i_3 - \dfrac{R_6(R_5+R_8)+R_5R_8}{(R_5+R_8)} i_4 + \dfrac{R_5}{R_5+R_8} v_s
\end{cases}
$$

And finally, in matrix form:

$$
\boxed{
\begin{pmatrix} \dfrac{dv_1}{dt} \\ \dfrac{dv_2}{dt} \\ \dfrac{di_3}{dt} \\ \dfrac{di_4}{dt} \end{pmatrix}
=
\begin{bmatrix}
-\dfrac{R_5+R_8+R_7}{C_1(R_5+R_8)R_7} & \dfrac{1}{C_1R_7} & 0 & \dfrac{R_5}{C_1(R_5+R_8)} \\
\dfrac{1}{C_2R_7} & -\dfrac{1}{C_2R_7} & -\dfrac{1}{C_2} & 0 \\
0 & \dfrac{1}{L_3} & -\dfrac{R_6}{L_3} & \dfrac{R_6}{L_3} \\
-\dfrac{R_5}{L_4(R_5+R_8)} & 0 & \dfrac{R_6}{L_4} & -\dfrac{R_6(R_5+R_8)+R_5R_8}{L_4(R_5+R_8)}
\end{bmatrix}
\begin{pmatrix} v_1 \\ v_2 \\ i_3 \\ i_4 \end{pmatrix}
+
\begin{bmatrix} \dfrac{1}{C_1(R_5+R_8)} \\ 0 \\ 0 \\ \dfrac{R_5}{L_4(R_5+R_8)} \end{bmatrix} v_s
}
$$

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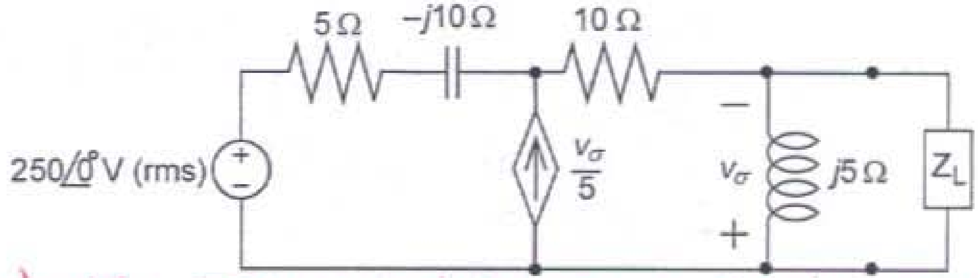
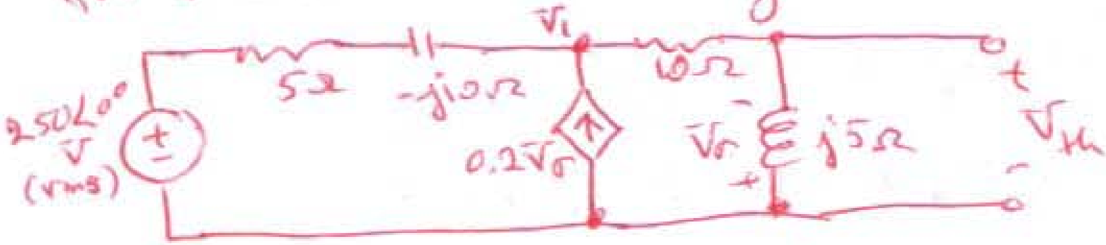
3. (25 points) The load impedance Z_L for this circuit is adjusted until maximum average power is delivered to Z_L.

a) Find the maximum average power delivered to Z_L.

b) What percentage of the total power developed in the circuit is delivered to Z_L?



(a) First let's get the Thevenin's equivalent circuit for the circuit seen by the load impedance:



$$\begin{cases} \dfrac{V_1 - 250}{5 - j10} - 0.2 V_\sigma + \dfrac{V_1}{10+j5} = 0 \\ V_\sigma = \dfrac{-j5}{10+j5} V_1 = \dfrac{-j V_1}{2+j1} \end{cases}$$

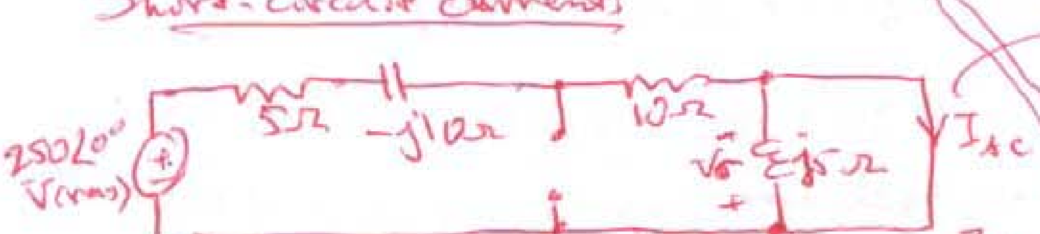
$$-0.2 V_\sigma = \frac{j0.2 V_1}{2+j1}$$

$$V_1 \left[\frac{1}{5-j10} + \frac{j0.2}{2+j1} + \frac{1}{10+j5} \right] = \frac{250}{5-j10}$$

$$\Rightarrow V_1 = 10(10+j5)$$

$$\therefore \underline{V_{th}} = \frac{j5}{10+j5} V_1 = j50 = 50\angle 90^\circ \text{ V (rms)}$$

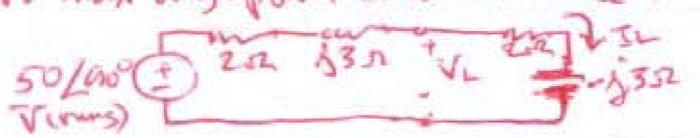
Short-circuit current:



$$I_{sc} = \frac{250\angle 0^\circ}{15 - j10} = \frac{50}{3-j2} \text{ A (rms)}$$

$$\underline{Z_{th}} = \frac{V_{th}}{I_{sc}} = \frac{j50}{50}(3-j2) = 2+j3\,\Omega$$

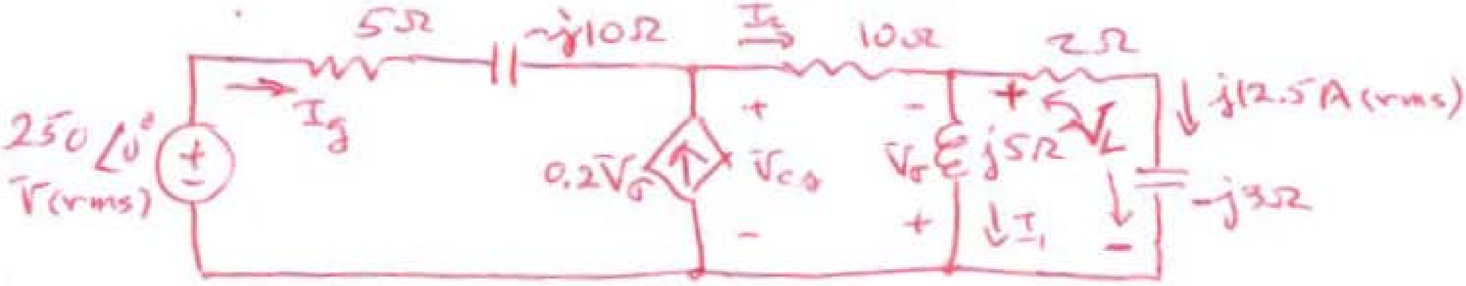
∴ For max avg. power delivered to Z_L:



$$I_L = \frac{50\angle 90^\circ}{4} = 12.5\angle 90^\circ \text{ A (rms)}$$

$$\Rightarrow \underline{P} = (12.5)^2(2) = \boxed{312.5 \text{ W}}$$

(b) In the original circuit, with the optimized Z_L we have:



$$V_L = (2-j3)(j12.5) = 37.5 + j25 \text{ V(rms)}$$

$$\Rightarrow I_1 = \frac{V_L}{j5} = \frac{37.5+j25}{j5} = 5 - j7.5 \text{ A (rms)}$$

$$\Rightarrow I_2 = I_1 + I_L = 5 - j7.5 + j12.5 = 5 + j5 \text{ A (rms)}$$

$$\Rightarrow V_{cs} = V_L + 10I_2 = 37.5 + j25 + 50 + j50 = 87.5 + j75 \text{ V(rms)}$$

$$\Rightarrow V_\sigma = -V_L = -37.5 - j25$$

$$\Rightarrow 0.2V_\sigma = -7.5 - j5 = I_{cs}$$

So complex power absorbed by the dependant source is

$$S_{cs} = V_{cs}(-I_{cs})^* = (87.5+j75)(7.5+j5)^*$$
$$= (87.5+j75)(7.5-j5) = (656.25+375) + j(562.5-437.5)$$
$$= \underline{1031.25 + j125 \text{ VA}}$$

Therefore, in this circuit the dependant source is absorbing 1031.25 W and 125 magnetizing vars.
(Only the indepent voltage source is developing power.)

$$I_g = -0.2V_\sigma + I_2 = 7.5 + j5 + 5 + j5 = 12.5 + j10 \text{ A (rms)}$$

$$S_g = (250)(-I_g)^* = (250)(-12.5 + j10) = -3125 + j2500 \text{ VA}$$

$$\therefore P_{dev} = 3125 \text{ W} \Rightarrow \% \text{ of devel. pwr deliv.} = \left(\frac{312.5}{3125}\right)(100) = \boxed{10\%}$$

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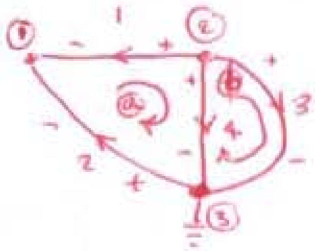
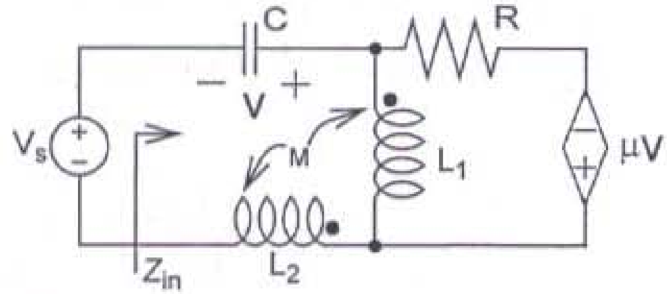
4. (25 points) This circuit is in the sinusoidal steady state with angular frequency ω.

a) Write the circuit's mesh equations in matrix form, with matrix and vector elements made explicit in terms of R, L, C, ω, etc., variables.

b) Determine the driving-point impedance Z_{in} when ω = 2 rad./sec. and R = 2 Ω, L_1 = 2 H, L_2 = 4H, |M| = 1H, C = 1/2 F, μ = 10.

(a)





$$-V_1 + V_2 + V_4 = 0$$
$$V_3 - V_4 = 0$$
} ← Σ of mesh KVL

$$\begin{cases} I_1 = -I_a \\ I_2 = I_a \\ I_3 = I_b \\ I_4 = I_a - I_b \end{cases}$$
← branch currents in terms of mesh currents

$$V_1 = \left(\frac{1}{j\omega C}\right) I_1 = -\frac{1}{j\omega C} I_a$$

$$V_2 = (j\omega L_2) I_2 + (j\omega M) I_4 - V_A = j\omega L_2 I_a + j\omega M (I_a - I_b) - V_A$$

$$V_3 = R I_3 - \mu V \quad (\text{where } V = V_1) \Rightarrow V_3 = R I_b + \mu\left(\frac{1}{j\omega C}\right) I_a$$

$$V_4 = (j\omega L_1) I_4 + (j\omega M) I_2 = j\omega L_1 (I_a - I_b) + j\omega M I_a$$

So (using Σ of mesh KVL):

$$\frac{1}{j\omega C} I_a + j\omega L_2 I_a + j\omega M (I_a - I_b) - V_A + j\omega L_1 (I_a - I_b) + j\omega M I_a = 0$$

&

$$R I_b + \mu\left(\frac{1}{j\omega C}\right) I_a - j\omega L_1 (I_a - I_b) - j\omega M I_a = 0$$

$$\Rightarrow \boxed{\begin{bmatrix} \frac{1}{j\omega C} + j\omega L_2 + j2\omega M + j\omega L_1 & -j\omega L_1 - j\omega M \\ \frac{\mu}{j\omega C} - j\omega L_1 - j\omega M & R + j\omega L_1 \end{bmatrix} \begin{pmatrix} I_a \\ I_b \end{pmatrix} = \begin{pmatrix} V_A \\ 0 \end{pmatrix}}$$

(b) Substituting specific numerical values into the mesh eqns.:

$$\begin{bmatrix} \frac{1}{j(2)(\frac{1}{2})} + j2(4+2\times1+2) & -j2(2+1) \\ \frac{10}{j(2)(\frac{1}{2})} - j2(2+1) & 2+j(2)(2) \end{bmatrix} \begin{pmatrix} I_a \\ I_b \end{pmatrix} = \begin{pmatrix} V_s \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} -j + j16 & -j6 \\ -j10 - j6 & 2+j4 \end{bmatrix} \begin{pmatrix} I_a \\ I_b \end{pmatrix} = \begin{pmatrix} V_s \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} j15 & -j6 \\ -j16 & 2+j4 \end{bmatrix} \begin{pmatrix} I_a \\ I_b \end{pmatrix} = \begin{pmatrix} V_s \\ 0 \end{pmatrix}$$

Using Cramer's rule: $I_a = \dfrac{\begin{vmatrix} V_s & -j6 \\ 0 & 2+j4 \end{vmatrix}}{\begin{vmatrix} j15 & -j6 \\ -j16 & 2+j4 \end{vmatrix}}$

$\cancel{Z_{eq}}$

$$\Rightarrow I_a = \frac{(2+j4)V_s}{j30 - 60 + 96} = \frac{(2+j4)V_s}{36+j30}$$

$$= \frac{(2+j4)(36-j30)}{36^2+30^2} V_s$$

$$= \frac{(72+120) + j(144-60)}{2196} V_s$$

$$= \frac{192+j84}{2196} V_s = \left(\frac{48+j21}{549}\right) V_s$$

Clearly, $Z_{in} = \dfrac{V_s}{I_a} = \dfrac{549}{48+j21} = \dfrac{549(48-j21)}{48^2+21^2}$

$$= \frac{26{,}352 - j11{,}529}{2304+441} = \frac{26{,}352}{2{,}745} - j\frac{11{,}529}{2{,}745}$$

$$\boxed{Z_{in} = 9.6 - j4.2}$$

Reference Sheet

Trigonometric Identities

$$\sin\theta = \cos(\theta - 90°) = \cos\left(\theta - \frac{\pi}{2}\right)$$

$$\cos\theta = \sin(\theta + 90°) = \sin\left(\theta + \frac{\pi}{2}\right)$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

$$\cos\theta + \cos\phi = 2\cos\frac{(\theta + \phi)}{2}\cos\frac{(\theta - \phi)}{2}$$

$$\cos\theta - \cos\phi = -2\sin\frac{(\theta + \phi)}{2}\sin\frac{(\theta - \phi)}{2}$$

$$\sin\theta + \sin\phi = 2\sin\frac{(\theta + \phi)}{2}\cos\frac{(\theta - \phi)}{2}$$

$$\sin\theta - \sin\phi = 2\cos\frac{(\theta + \phi)}{2}\sin\frac{(\theta - \phi)}{2}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

$$A\cos\theta + B\sin\theta = (A^2 + B^2)^{1/2}\cos(\theta - \operatorname{atan}(B/A))$$

Complex Arithmetic

$$\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$$

$$\mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$$

$$\mathrm{Re}(z_1 z_2) = \mathrm{Re}(z_1)\mathrm{Re}(z_2) - \mathrm{Im}(z_1)\mathrm{Im}(z_2)$$

$$\mathrm{Im}(z_1 z_2) = \mathrm{Re}(z_1)\mathrm{Im}(z_2) + \mathrm{Im}(z_1)\mathrm{Re}(z_2)$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$Z = X + jY \Rightarrow Z = Re^{j\theta}$, where $R = (X^2 + Y^2)^{1/2}$ and $\theta = \mathrm{atan}(Y/X)$

$Z = Re^{j\theta} \Rightarrow Z = X + jY$ where $X = R\cos\theta$ and $Y = R\sin\theta$

$$|X + jY| = (X^2 + Y^2)^{1/2}$$
$$\angle(X + jY) = \mathrm{atan}(Y/X)$$

$$|Z_1 Z_2| = |Z_1||Z_2|;\ \angle(Z_1 Z_2) = \angle Z_1 + \angle Z_2$$

$$|1/Z| = 1/|Z|\ ;\ \angle(1/Z) = -\angle Z$$

$$(X + jY)^* = X - jY$$

$$(Re^{j\theta})^* = Re^{-j\theta}$$

$$(Z_1 Z_2)^* = (Z_1^*)(Z_2^*); \qquad (Z_1/Z_2)^* = (Z_1^*)/(Z_2^*); \quad (Z_1 \pm Z_2)^* = (Z_1^*) \pm (Z_2^*)$$