EE 10, Fall 2014, Midterm Exam – November 19, 2014

Instructions: This exam booklet consists of four problems, blank sheets for the solutions, reference sheets with mathematical identities, and additional blank sheets. Please follow these instructions while answering your exam:

1. **Write your name and student identification number below.**
2. **Write the names of students to your left and right as well.**
3. **You have 1 hour 45 minutes to finish your exam.**
4. **Write your solutions in the provided blank sheets after each problem.**
5. **The sheets marked "Scratch..." will NOT be graded. These sheets are provided for your rough calculations only.**
6. **Write your solutions clearly. You may box in your final answer. Illegible solutions will NOT be graded.**
7. **Be brief.**
8. **Open Book & open notes only. NO homework solution!**

NAME: Solution Key

STUDENT ID: ______________________

NAMES OF ADJACENT STUDENTS:

LEFT: ______________________

RIGHT: ______________________

Problem 1: Assume that the circuit achieved steady state before the switch moves to position 2 and that L_2 has no energy stored on it. **Ignore the mutual inductance for parts (a) and (b).**

Problem #1:

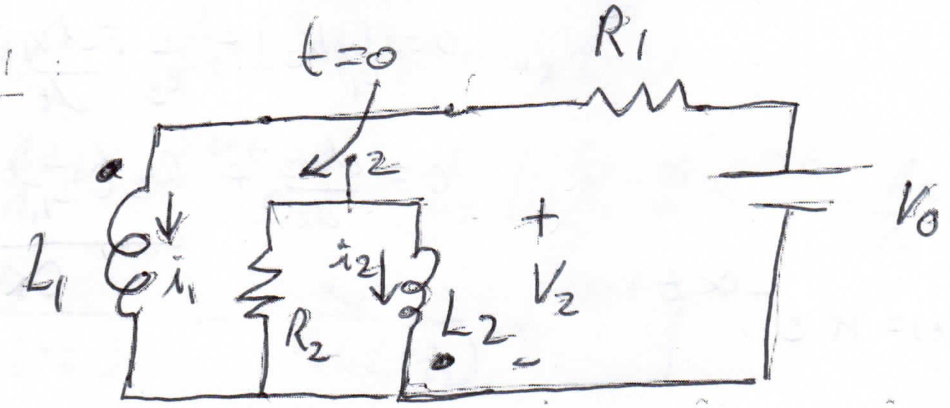


Figure 1.

a. Determine the currents in the inductors at time t = 0+ , i.e. $i_1(0+)$ & $i_2(0+)$.

b. Determine an expression for the voltage $i_2(t)$ for all time $t \geq 0$. Draw a rough sketch of the waveform.

Assume a mutual inductance of M for the remaining parts with dots as shown in the figure.

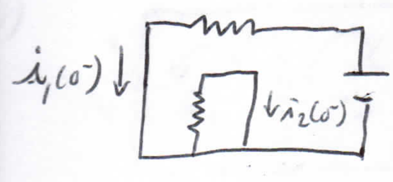
c. Write mesh equations for $t \geq 0$. Define the meshes such that i_1 and i_2 are your mesh unknowns.

d. What are the currents in the inductors at time t = 0+, i.e. $i_1(0+)$ & $i_2(0+)$? Is your answer different from (a)? Explain.

e. What is the value of first derivative of $i_2(t)$ at time t = 0+? Explain. Assume $L_1 = L_2$ for simplicity.

(6 + 14 + 6 + 6 + 8 = 40 points)

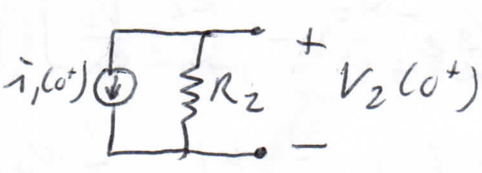
a) $t < 0$



$i_1(0^-) = \frac{V_o}{R_1}$

$i_2(0^-) = 0$

$t \geq 0$



$V_2(0^+) = -i(0^+) R_2$

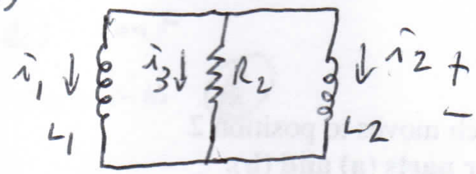
$i_1(0^+) = i_1(0^-) = \frac{V_o}{R_1}$

$i_2(0^+) = i_2(0^-) = 0$

$V_2(0^+) = -\frac{R_2}{R_1} V_o$

b)

KCL



$$(1)\quad 0 = \frac{1}{L_1}\int V_2\,dt + \frac{V_2}{R_2} + \frac{1}{L_2}\int V_2\,dt$$

$$0 = \frac{1}{L_1}V_2 + \frac{1}{R_2}\cdot\frac{dV_2}{dt} + \frac{1}{L_2}V_2$$

$$0 = \frac{dV_2}{dt} + \underbrace{R_2\left(\frac{1}{L_1}+\frac{1}{L_2}\right)}_{\alpha}V_2$$

$$V_2(t) = K e^{-\alpha t}$$

- finding K

$$V_2(0^+) = K = -\frac{R_2}{R_1}V_0 \quad , \quad V_2(t) = -\frac{R_2}{R_1}V_0 e^{-\alpha t}$$

$$i_1(t) = \frac{1}{L_1}\int_{-\infty}^{t} V_2(t)\,dt = \frac{1}{L_1}\int_0^t V_2(t)\,dt + i_1(0^+)$$

from (1)

$$\underbrace{\frac{1}{L_2}\int V_2\,dt}_{i_2(t)} = -\frac{1}{L_1}\int V_2(t)\,dt - \frac{V_2}{R_2}$$

$$i_2(t) = -\frac{1}{L_1}\int_0^t -\frac{R_2}{R_1}V_0 e^{-\alpha t}\,dt + \frac{-V_0}{R_1} + \frac{V_0}{R_1}e^{-\alpha t}$$

$$= -\frac{1}{L_1}\left[\frac{R_2}{R_1}\cdot\frac{V_0}{\alpha}e^{-\alpha t} - \frac{R_2}{R_1}\cdot\frac{V_0}{\alpha}\right] - \frac{V_0}{R_1} + \frac{V_0}{R_1}e^{-\alpha t}$$

$$= \frac{1}{L_1}\cdot\frac{R_2}{R_1}\cdot\frac{V_0}{\alpha}\left[1-e^{-\alpha t}\right] + -\frac{V_0}{R_1}\left[1-e^{-\alpha t}\right]$$

$$= \left[1-e^{-\alpha t}\right]\left[\frac{1}{L}\cdot\frac{R_2}{R_1}\cdot\frac{V_0}{R_2\left(\frac{1}{L_1}+\frac{1}{L_2}\right)} - \frac{V_0}{R_1}\right]$$

$$= \left[1-e^{-\alpha t}\right]\left[\frac{V_0}{R_1}\cdot\frac{1}{1+\frac{L_1}{L_2}} - \frac{V_0}{R_1}\right]$$

$$= \frac{V_0}{R_1}\left[1-e^{-\alpha t}\right]\left[\frac{L_2}{L_1+L_2} - 1\right]$$

$$i_2(t) = \frac{V_0}{R_1}\left[1-e^{-\alpha t}\right]\left[\frac{-L_1}{L_1+L_2}\right]$$

to check

$$i_2(t) = \frac{1}{L_2}\int_{-\infty}^{t} v_2\,dt = \frac{1}{L_2}\int_0^t v_2\,dt + \cancel{i_2(0^+)}^{\,0}$$

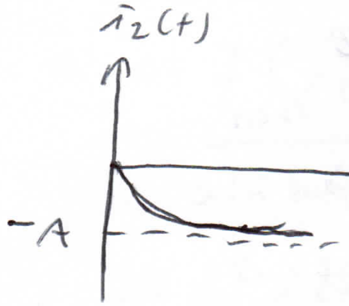
$$i_2(t) = \frac{1}{L_2}\int_0^t -\frac{R_2}{R_1}V_0 e^{-\alpha t}\,dt = \frac{1}{L_2}\left[\frac{R_2}{R_1}\cdot\frac{V_0}{\alpha}e^{-\alpha t} - \frac{R_2}{R_1}\cdot\frac{V_0}{\alpha}\right]$$

$$= -\frac{1}{L_2}\cdot\frac{V_0}{R_1}\left[\frac{R_2}{R_2\left(\frac{1}{L_1}+\frac{1}{L_2}\right)} - \frac{R_2}{R_2\left(\frac{1}{L_1}+\frac{1}{L_2}\right)}e^{-\alpha t}\right]$$

$$= -\frac{V_0}{R_1}\cdot\frac{1}{\left(\frac{L_2}{L_1}+1\right)}\left[1-e^{-\alpha t}\right]$$

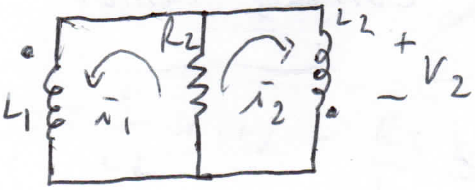
$$\boxed{i_2(t) = -\frac{V_0}{R_1}\left[\frac{L_1}{L_2+L_1}\right]\left[1-e^{-\alpha t}\right] \quad \checkmark}$$

So waveform will look like



$$A = \frac{V_0}{R_1}\left[\frac{L_1}{L_2+L_1}\right]$$

c)



A $\quad 0 = L_1\frac{di_1}{dt} + R_2(i_1+i_2) - M\frac{di_2}{dt}$

B $\quad 0 = L_2\frac{di_2}{dt} + R_2(i_2+i_1) - M\frac{di_1}{dt}$

d) b/c the current cannot change inst. through an inductor by inspection you can see that when $t < 0$

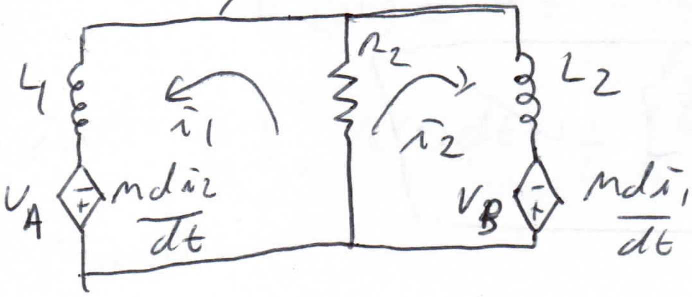
$$i_1(0^-) = \frac{V_0}{R_1}, \quad i_2(0^-) = 0$$

$\therefore$ $i_1(0^-) = i_1(0^+)$ & $i_2(0^-) = i_2(0^+)$

and from part (a) $V_2(0^+) = -\frac{R_2}{R_1} V_0$

by the time the switch moves the circuit will already be in S.S. so the inductors are shorts $(t=0^-)$ the $\frac{di}{dt}$ terms will go to zero.

Another way to look at it



The coupling is represented by the dependent sources

looking at eqs. (A) & (B) if i_1 were to increase, V_B increases and thus i_2 increases, looking at this V_1 will then increase and i_1 will get larger than before. This is positive feed back and the 2 currents will go to infinity which is not possible in this physical system, so the currents cannot change instantaneously.

e) adding (A) & (B) from part c) $(L_1 = L_2)$

$$0 = (L-M)i_1' + (L-M)i_2' + 2R_2(i_1 + i_2)$$

@ $t = 0^+$ $\quad i_2(0^+) = 0 \quad i_1(0^+) = \frac{V_0}{R_1}$

$$(L-M)(i_1' + i_2') = -2\frac{R_2}{R}V_0$$

$$i_1' = \frac{-2\frac{R_2}{R_1}V_0}{L-M} - i_2'$$

e) cont

sub into (B)

$$0 = L i_2'(0^+) + R_2 \cdot \frac{V_0}{R_1} - M \left[\frac{-2 \frac{R_2}{R_1} V_0}{L-M} - i_2'(0^+) \right]$$

$$-(L+M) i_2'(0^+) = \frac{R_2}{R_1} V_0 + \frac{2M \frac{R_2}{R_1} V_0}{L-M}$$

$$i_2'(0^+) = \frac{-\frac{R_2}{R_1} V_0}{L+M} \left[1 + \frac{2M}{L-M} \right]$$

$$= \frac{-\frac{R_2}{R_1} V_0}{\cancel{L+M}} \left[\frac{\cancel{L+M}}{L-M} \right]$$

$$\boxed{i_2'(0^+) = -\frac{R_2}{R_1} V_0 \left[\frac{1}{L-M} \right]}$$

Problem 2: Refer to Figure 2 for this problem. The inductor has no initial energy.

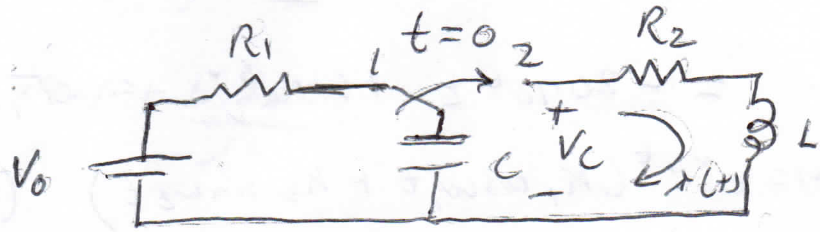


Figure 2.

a. Derive the differential equation that governs the current i(t) for time $t \geq 0$.

b. What is the characteristic equation?

For the following assume $R_1 = 5\ k\Omega$, $R_2 = 16\ \Omega$, $L = 400$ nH, $C = 4$ nF, and $V_0 = 10$ Volts.

c. Derive an expression for the voltage across the capacitor, $V_C(t)$ for $t \geq 0$. Assume that the circuit achieved steady state before $t = 0$ i.e. assume that the circuit came into being at $t=-\infty$. Draw a rough sketch of the waveform.

d. Suppose that the circuit came into being only at $t = -20\ \mu s$ rather than at $t=-\infty$ with none of the components energized prior to that. Derive a new expression for $V_C(t)$ for $t \geq 0$. How does it compare to your answer in part (c)?

(10 + 5 + 10 + 10 = 35 points)

a) $$0 = \underbrace{\frac{1}{C}\int i(t)\,dt}_{-V_c(t) \text{ by sign convention}} + R_2 i(t) + L\frac{di(t)}{dt} \quad (1)$$

$$0 = \frac{1}{C} i(t) + R_2 \frac{di(t)}{dt} + L\frac{d^2 i(t)}{dt^2} \quad (2)$$

$$0 = \frac{1}{CL} i(t) + \frac{R_2}{L}\frac{di(t)}{dt} + \frac{d^2 i(t)}{dt^2} \quad (3)$$

b) $$0 = \lambda^2 + \frac{R_2}{L}\lambda + \frac{1}{CL}$$

c)

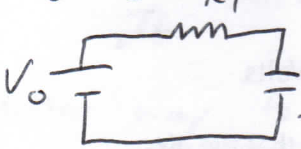
$$\lambda_1, \lambda_2 = \frac{-\frac{R_2}{L} \pm \sqrt{\left(\frac{R_2}{L}\right)^2 - \frac{4}{CL}}}{2}$$

$$= -20 \cdot 10^6 \pm 15 \cdot 10^6 j = \sigma_1 \pm j\omega_1$$

$$i(t) = e^{\sigma_1 t}\left(K_1 \cos\omega_1 t + K_2 \sin\omega_2 t\right) \quad (4)$$

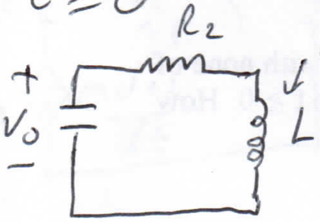
- Finding $i(0^+)$, $\frac{di(0^+)}{dt}$, $V_C(0^+)$

$t < 0$



$V_C(0^-) = V_0$ $\qquad V_C(0^-) = V_C(0^+)$

$t \geq 0$



acts as open @ $t = 0^+$

$i(0^-) = i(0^+) = 0$

using (1)

$$0 = \underbrace{\frac{1}{C}\int i(0^+)\,dt}_{-V_C(0^+)} + \cancel{R_2 i(0^+)}^{\,0} + \frac{L\,di(0^+)}{dt}$$

$$\frac{V_C(0^+)}{L} = \frac{di(0^+)}{dt}$$

$$i(0^+) = 0 = K_1 \rightarrow i(t) = e^{\sigma_1 t}\left(K_2 \sin\omega_1 t\right)$$

$$i'(t) = \sigma_1 e^{\sigma_1 t} K_2 \sin\omega_1 t + K_2 \omega_1 e^{\sigma_1 t} \cos\omega_1 t$$

$$i'(0^+) = \frac{V_0}{L} = K_2 \omega_1 \rightarrow K_2 = \frac{V_0}{L\omega_1}$$

$i(t) = e^{\sigma_1 t}\left(\frac{V_o}{\omega_1 L}\sin\omega_1 t\right)$, to find $V_c(t)$ we use ①

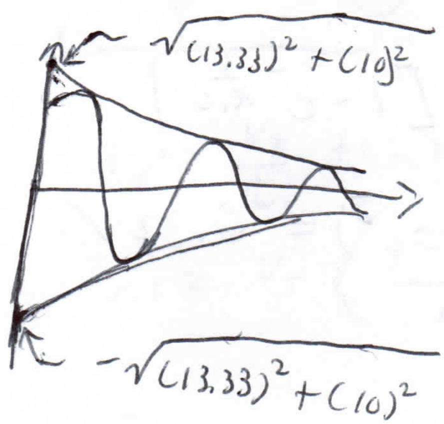
$$\underbrace{-\frac{1}{C}\int i(t)\,dt}_{V_c(t)} = R_2 i(t) + \frac{L\,di(t)}{dt}$$

$$\boxed{\begin{aligned} V_c(t) &= R_2\left[e^{\sigma_1 t}\sin\omega_1 t\right]\cdot\frac{V_o}{\omega_1 L} + L\left[\sigma_1 e^{\sigma_1 t}\sin\omega_1 t + \omega_1 e^{\sigma_1 t}\cos\omega_1 t\right]\cdot\frac{V_o}{\omega_1 L} \\ V_c(t) &= e^{\sigma_1 t}\left[13.33\sin\omega_1 t + 10\cos\omega_1 t\right] \quad t \geq 0 \end{aligned}}$$

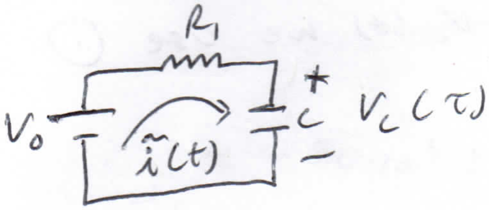
note we could have just used $V_c(t) = -\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt$

remembering that $\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt = \frac{1}{C}\int_{0}^{t} i(t)\,dt + V_c(0^+)$

The wave form will look like (roughly)



d) for $t \le 0$ the capacitor charges only for $20\mu s$ before the switch moves. So $V_c(0^+)$ changes



$\tau = t + 20\mu s$ → new reference

$V_c(t = -20\mu s) = 0 = V_c(\tau = 0)$

$\tilde{i}(0 = \tau) = \frac{V_0}{R_1}$

$$\tilde{i}(\tau) R_1 + \frac{1}{C}\int \tilde{i}(\tau)\, d\tau = V_0$$

$$R_1 \frac{d}{d\tau}\tilde{i}(\tau) + \frac{1}{C}\tilde{i}(\tau) = 0$$

$$\frac{d\tilde{i}(\tau)}{d\tau} + \frac{1}{R_1 C}\tilde{i}(\tau) = 0$$

$$\tilde{i}(\tau) = K_0 e^{-\frac{\tau}{R_1 C}} \longrightarrow \tilde{i}(0) = \frac{V_0}{R_1} = K_0$$

$$\frac{1}{C}\int_{-\infty}^{\tau} \tilde{i}(\tau)\, d\tau = \frac{1}{C}\int_0^{\tau} \tilde{i}(\tau)\, d\tau + \underbrace{V_c(\tau = 0)}_{0}$$

$$V_c(\tau) = \frac{1}{C}\left[-V_0 C e^{-\frac{\tau}{R_1 C}}\right]\Big|_0^{\tau} = V_0\left[1 - e^{-\frac{\tau}{R_1 C}}\right]$$

$$V_c(\tau = 20\mu s) = V_0\left(1 - e^{-\frac{20\mu s}{R_1 C}}\right] = V_c(t = 0^-) = 6.32\, V$$

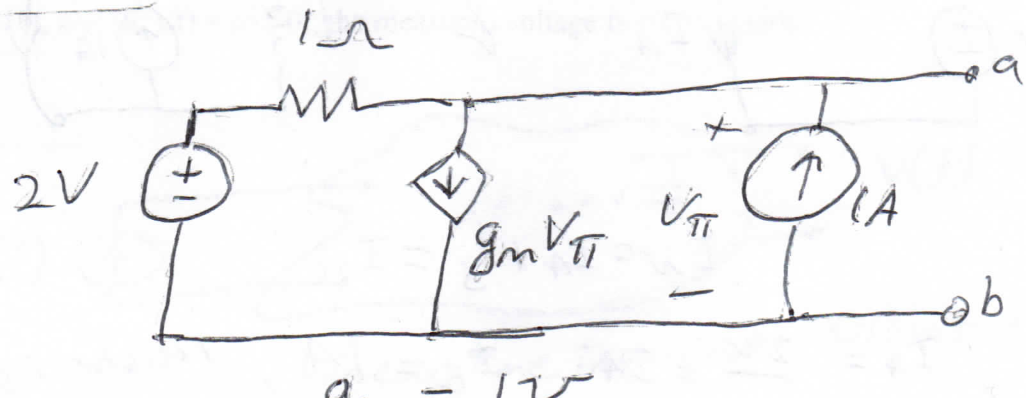
$R_1 C = 20\mu s$

$$\boxed{V_c(t = 0^-) = V_c(t = 0^+) = 6.32\, V}$$

expression for $V_c(t)$ $t \ge 0$

$$\boxed{V_c(t) = \left[8.43 \sin\omega_1 t + 6.32 \cos\omega_1 t\right] e^{\sigma_1 t}}$$

Problem 3: Derive the Norton's equivalent of this circuit.

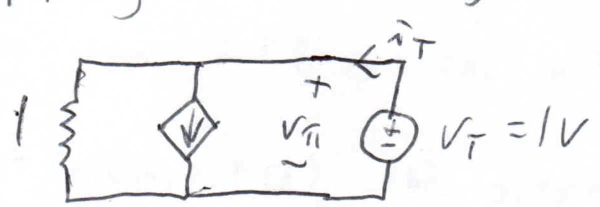


$g_m = 1℧$

Figure 3.

(10 points)

Finding R_N (R_{Th})

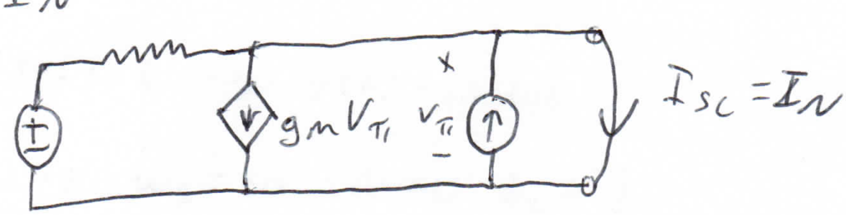


$V_T = V_\pi$ $\qquad \frac{V_T}{i_T} = R_{Th}$

$$i_T = \frac{V_T}{R} + g_m V_\pi \rightarrow i_T = V_T\left(\frac{1}{R} + g_m\right)$$

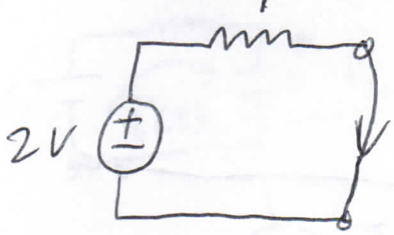
$$i_T = V_T\left(\frac{1+Rg_m}{R}\right) \rightarrow \frac{V_T}{i_T} = \frac{R}{1+Rg_m} = R_{Th} = \frac{1}{2}\,\Omega$$

finding I_N

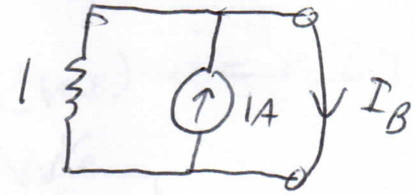


b/c of short circuit $V_\pi = 0$, so $g_m V_\pi = 0$

by super posistion



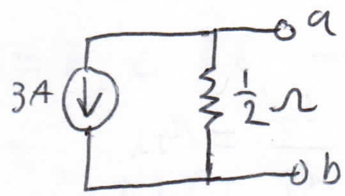
&



$$I_N = I_A + I_B = I_{sc}$$

$$I_A = \frac{2V}{1\Omega} = 2A \qquad I_B = 1A, \text{ resistor is shorted}$$

$$I_N = 3A$$



The polarity of the current source should be reversed. This is because when shorting out terminals a-b the current will flow from a to b. Or when looking at the open circuit voltage, the voltage is positive across a to b.

Problem 4: Refer to Figure 4 for this problem. There are no independent sources inside the linear time-invariant network N. The only independent source is the current source, i(t), shown in the figure. The network components have no stored energy. The voltage across some capacitor in the network is measured using an ideal voltmeter. It was found that for i(t) = 2cos10t, the measured voltage is v(t) = 8sin10t, and for i(t) = cos30t, the measured voltage is v(t) = sin30t.

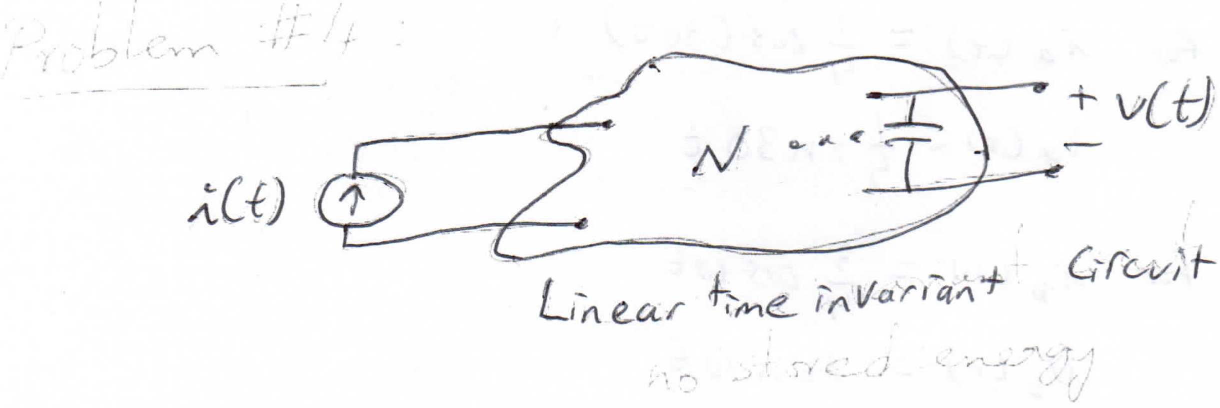


Figure 4.

Using the information provided, what would the voltage be for $i(t) = \cos^3 10t$? Explain.

Hint: $cos^3 A = \frac{1}{4}cos(3A) + \frac{3}{4}cosA$.

(15 points)

For an LTI system with a general sinusoidal input $A\cos(w_0 t + \theta)$, the output is $A \cdot K_0 \cos(w_0 t + \phi_0 + \theta)$

where K_0, & ϕ_0 is dependent on w_0 but are constants

So for $i(t) = 2\cos 10t \rightarrow V(t) = 8\sin 10t$

$A = 2 \quad w_1 = 10 \quad \theta = 0 \quad \phi_1 = \frac{\pi}{2} \quad k_1 = 4$, $\cos(x - \frac{\pi}{2}) = \sin x$

for $i(t) = \cos 30t \rightarrow V(t) = \sin 30t$

$A = 1 \quad k_2 = 1 \quad w_2 = 30 \quad \theta = 0 \quad \phi_2 = \frac{\pi}{2}$

so for $i(t) = \frac{1}{4}\cos(30t) + \frac{3}{4}\cos 10t$ we can

use superposition and use $i_a(t) = \frac{1}{4}\cos(30t)$

& $i_b(t) = \frac{3}{4}\cos 10t$

so for $i_a(t) = \frac{1}{4}\cos(30t)$

$$V_a(t) = \frac{1}{4}\sin 30t$$

and for $i_b(t) = \frac{3}{4}\cos 10t$

$$V_b(t) = 3\sin 10t$$

so $v(t) = V_a(t) + V_b(t) = \frac{1}{4}\sin 30t + 3\sin 10t$