ECE10 **Midterm Exam** **Fall 2018**

Total of 3 questions, 100 minutes.

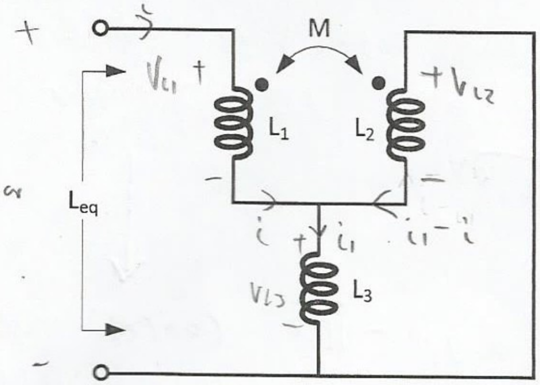
P1 (30)	28
P2 (30)	30
P3 (40)	40
Total (100)	98

1. For the circuit shown below,
   a. Find the equivalent inductance, L_{eq}.
   b. Show that L_{eq} is always positive.

$0 \le k = \frac{M}{\sqrt{L_1 L_2}} \le 1$

$M \le \sqrt{L_1 L_2}$

$M^2 \le L_1 L_2$



$i + i_1 - i_1 = i$ ✓

a)
$$\begin{cases} V_{L_1} = L_1 \frac{di}{dt} + M\frac{d(i_1 - i)}{dt} \\ V_{L_2} = L_2 \frac{d(i_1-i)}{dt} + M\frac{di}{dt} = -V_{L_3} = -L_3 \frac{di_1}{dt} \end{cases}$$

$$\frac{L_2 di_1}{dt} - \frac{L_2 di}{dt} + \frac{M di}{dt} = -L_3 \frac{di_1}{dt}$$

$$(L_2 - M)\frac{di}{dt} = (L_3 + L_2)\frac{di_1}{dt}$$

$$\frac{di_1}{dt} = \frac{L_2 - M}{L_3 + L_2} \frac{di}{dt}$$

$$V_{eq} = L_{eq}\frac{di}{dt} = V_{L_1} + V_{L_3}$$

$$= L_1 \frac{di}{dt} + M\frac{d(i_1 - i)}{dt} + L_3\frac{di_1}{dt}$$

$$= (L_1 - M)\frac{di}{dt} + (M + L_3)\frac{di_1}{dt}$$

$$L_{eq}\cancel{\frac{di}{dt}} = (L_1 - M)\cancel{\frac{di}{dt}} + (M + L_3)\left(\frac{L_2 - M}{L_3 + L_2}\right)\cancel{\frac{di}{dt}}$$

$$L_{eq} = \frac{(L_1 - M)(L_2 + L_3) + (L_3 + M)(L_2 - M)}{L_2 + L_3}$$

$$L_{eq} = \frac{L_1 L_2 + L_1 L_3 - \cancel{L_2 M} - L_3 M + L_2 L_3 - L_3 M + \cancel{L_2 M} - M^2}{L_2 + L_3}$$

$$\boxed{L_{eq} = \frac{L_1 L_2 + L_2 L_3 + L_1 L_3 - 2 L_3 M - M^2}{L_2 + L_3}}$$

✓ $\rightarrow L_{eq} = L_1 + \frac{L_2 L_3}{L_2 + L_3} - \left(\frac{M^2 + 2L_3 M}{L_2 + L_3}\right) > 0$

$L_1 L_3$, and $L_2 + L_3$ are all positive because inductances are positive. We know that

$0 \le M^2 \le L_1 L_2$ from the definition of the coupling factor

$[0 \le |M| \le +\sqrt{L_1 L_2}]$

simplify L_{eq}:

$$L_{eq} = L_1 + \frac{L_2 L_3}{L_2 + L_3} - \left(\frac{M^2 + 2ML_3}{L_2 + L_3}\right) \overset{?}{>} 0$$

$$L_1 + \frac{L_2 L_3}{L_2 + L_3} > \frac{M^2 + 2ML_3}{L_2 + L_3}$$

$$L_1 L_2 + L_1 L_3 + L_2 L_3 > M^2 + 2ML_3$$

$$L_1 L_2 + L_1 L_3 + L_2 L_3 + L_3^2 > M^2 + 2ML_3 + L_3^2$$

$$L_1 L_2 + L_1 L_3 + L_2 L_3 + L_3^2 > (M + L_3)^2$$

$$-\sqrt{L_1 L_2 + L_1 L_3 + L_2 L_3 + L_3^2} < M + L_3 < \sqrt{L_1 L_2 + L_1 L_3 + L_2 L_3 + L_3^2}$$

$$-L_3 - \sqrt{L_1 L_2 + L_1 L_3 + L_2 L_3 + L_3^2} < M < \sqrt{L_1 L_2 + L_1 L_3 + L_2 L_3 + L_3^2} - L_3$$

$-(L_3 + \sqrt{L_1 L_2 + L_1 L_3 + L_2 L_3 + L_3^2})$

$< M <$

$\sqrt{L_1 L_2 + L_1 L_3 + L_2 L_3 + L_3^2} - \sqrt{L_3^2}$

↓

See back side for analysis of inequality

$\downarrow$

(i) because M is defined to be $M > -\sqrt{L_1 L_2}$, the left side of the inequality holds true. The other factors than $\sqrt{L_1 L_2}$ on the left only make the quantity increasingly negative. Therefore, the left side of the inequality holds true, as M will always be greater than $-L_3 - \sqrt{L_1 L_2 + L_2 L_3 + L_1 L_3 + L_3^2}$

$\downarrow$

(ii) The factors $\sqrt{L_3^2}$ and $-\sqrt{L_3^2}$ cancel out in their effect on the right side of the inequality. Effectively, we want to prove that $M < \sqrt{L_1 L_2 + L_2 L_3 + L_1 L_3 + L_3^2} - \sqrt{L_3^2}$ given that $M < +\sqrt{L_1 L_2}$. So, we need to prove that

$$\sqrt{L_1 L_2 + L_2 L_3 + L_1 L_3 + L_3^2} - \sqrt{L_3^2} \geq \sqrt{L_1 L_2}$$

$$L_1 L_2 + L_2 L_3 + L_1 L_3 + L_3^2 - 2\sqrt{L_1 L_2 L_3^2 + L_2 L_3^3 + L_1 L_3^3 + L_3^4} + L_3^2 \geq L_1 L_2$$

$$L_2 L_3 + L_1 L_3 + L_3^2 - 2\sqrt{L_1 L_2 L_3^2 + L_2 L_3^3 + L_1 L_3^3 + L_3^4} + L_3^2 \geq 0$$

The contribution to this inequality from L_1 & L_2 is insignificant, because the leftmost term will always exceed the center one in terms of L_1 & L_2's contributions, due to the power of each term (1 vs. $\frac{1}{2}$).

Therefore, because the contributions of L_3 cancel out, we can say that $M < \sqrt{L_1 L_2 + L_2 L_3 + L_1 L_3 + L_3^2} - \sqrt{L_3^2}$.

$0 \leq M \leq \sqrt{L_1 L_2}$

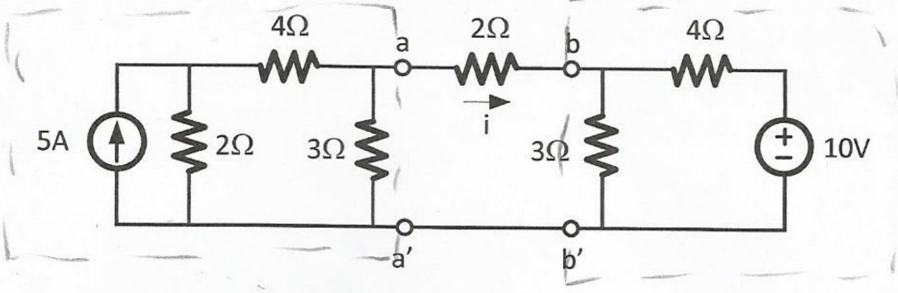
Hence, $\sqrt{L_1 L_2} \leq M \leq \sqrt{L_1 L_2}$ lies within the range of $-L_3 - \sqrt{L_1 L_2 + L_2 L_3 + L_1 L_3 + L_3^2} < M < \sqrt{L_1 L_2 + L_2 L_3 + L_1 L_3 + L_3^2} - \sqrt{L_3^2}$, so we have proven that $L_{eq} > 0$

$\rightarrow 2$

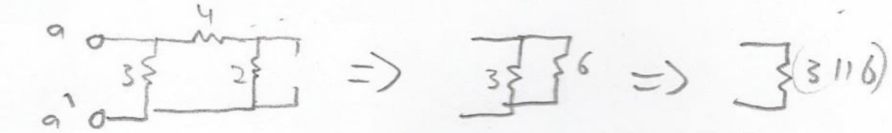
$P_R = \frac{2}{36}$

$\boxed{P_R = \frac{1}{18} W}$

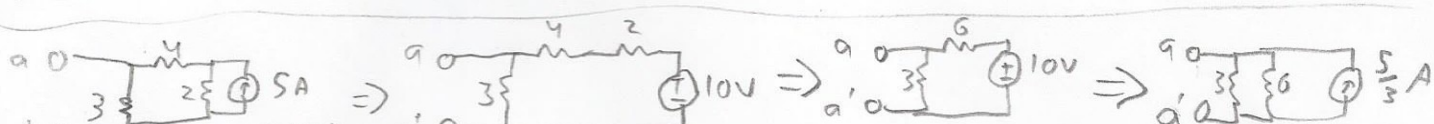
2. In the circuit below,
   a. Find the Thevenin equivalent at left side of $a - a'$ nodes.
   b. Find the Thevenin equivalent at the right side of $b - b'$ nodes
   c. Find the current i by constructing an equivalent circuit using part a and b Thevenin circuits.
   d. Calculate the power dissipated in the middle 2Ω resistor. $P_R = i^2 R = 2i^2$



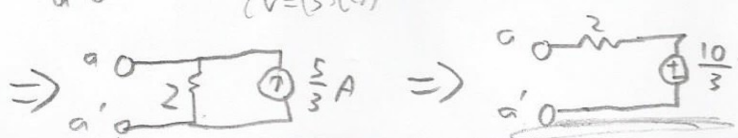
a)



$R_{eq} = \frac{(3)(6)}{3+6} = \frac{18}{9} = 2\,\Omega$



$(v = (5)(2))$ $(i = \frac{10}{6})$

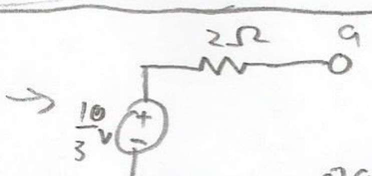


$(v = (\frac{5}{3})(2))$

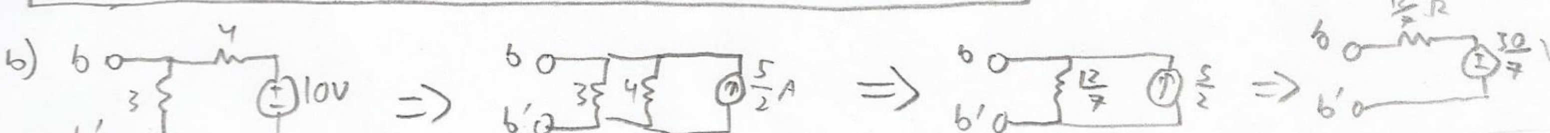
Thevenin Equivalent: (Left)

$R_{Th} = 2\,\Omega$

$V_{Th} = \frac{10}{3}$ V



b)

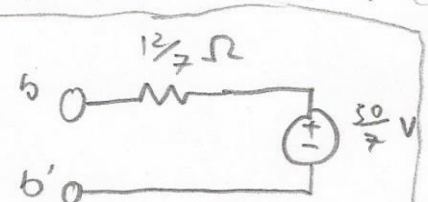


$(i = \frac{10}{4})$ $R = \frac{(3)(4)}{3+4}$ $v = (\frac{5}{2})(\frac{12}{7})$

Thevenin Equivalent: (Right)

$R_{Th} = \frac{12}{7}\,\Omega$

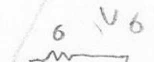
$V_{Th} = \frac{30}{7}$ V



Check:

$V_b = \frac{3}{7}(10)$

$V_b = \frac{30}{7}$

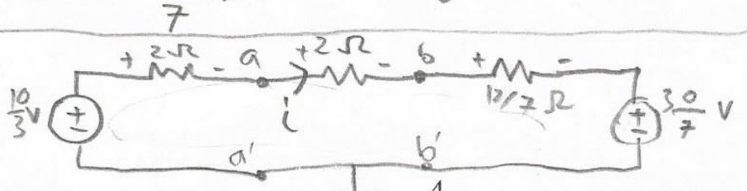


$V_a = \frac{3}{9}(10)$

$V_a = \frac{10}{3}$

$V_R = V_a - V_b = \frac{10}{3} - \frac{30}{7} = -\frac{20}{21}$

c) Equivalent Circuit:

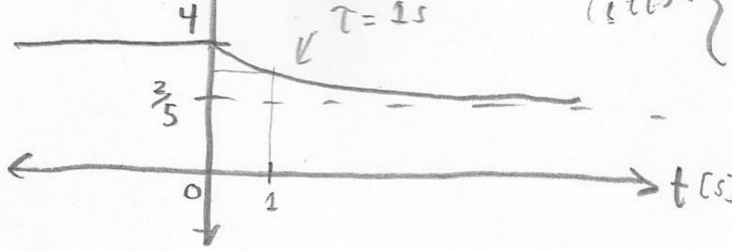


KVL: $0 = \frac{10}{3} - (2)(i) - (2)(i) - (\frac{12}{7})(i) - \frac{30}{7} = \frac{70-90}{21} - (\frac{28+12}{7})i = -(\frac{20}{21} + \frac{40}{7}i)$

$\frac{40}{7}i = -\frac{20}{21} \rightarrow \boxed{i = -\frac{1}{6} A}$

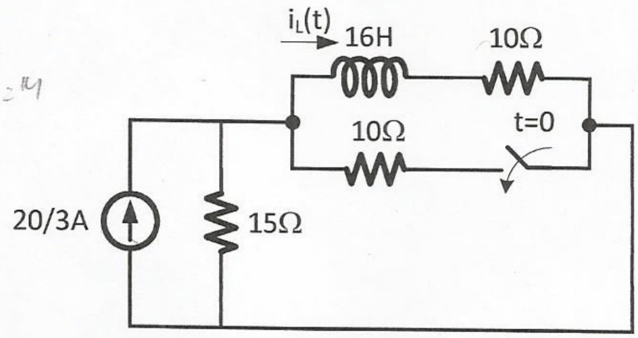
(see top for pt. d)

$\left(\frac{5}{2}+\frac{3}{2}e^{-t}\right.$ for $t>0$

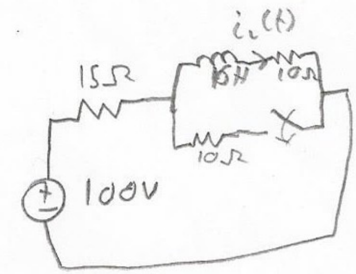


3. The circuit below has been idle for a long time (switch is open). At $t = 0$, the switch is closed.
   a. Find the inductor current right before and after the switch is closed ($i_L(0^-)$, and $i_L(0^+)$).
   b. Find and plot $i_L(t)$.

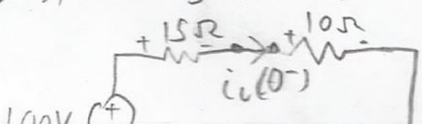
$\frac{15}{25}\cdot\frac{20}{3} = \frac{5}{5}\cdot\frac{4}{1} = 4$



$\rightarrow$



a) Steady state : $t = 0^-$, inductor is shorted ...



$$KVL : 100 - 15[i_L(0^-)] - 10[i_L(0^-)] = 0$$

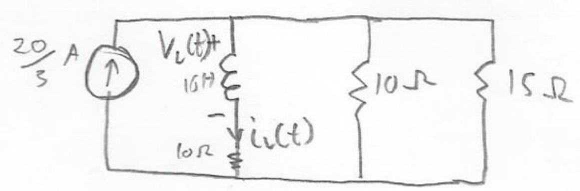
$$100 = 25[i_L(0^-)]$$

$$\boxed{i_L(0^-) = 4A}$$

Because there is no impulse/infinit current when the switch is closed, we have $\boxed{i_L(0^+) = i_L(0^-) = 4A}$

b) $t = 0^+$: inductor has full charge ... acts as a current source

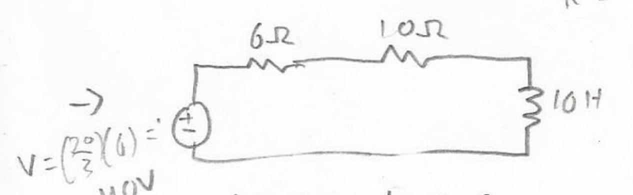
$$V_L(t) = L\frac{di(t)}{dt} \rightarrow i_L(t) = \frac{1}{L}\int_{0^+}^{t} V_L(t)dt$$



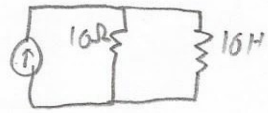
$\rightarrow$

$$R = \frac{(10)(15)}{10+15} = \frac{150}{25} = \frac{30}{5} = 6\Omega$$

$\rightarrow$



$\rightarrow i = \frac{40}{16} = \frac{5}{2}A$



$$\left\{\begin{array}{l}\tau = \frac{L}{R} = \frac{16H}{16\Omega} = 1s \\ i_L(0^+) = i_I = 4A \\ i_L(\infty) = i_F = \frac{5}{2}A\end{array}\right.$$

$$i_L(t) = i_F + (i_I - i_F)e^{-t/\tau}$$

$$i_L(t) = \frac{5}{2} + \left(4 - \frac{5}{2}\right)e^{-t/1}$$

$$\boxed{i_L(t) = \frac{5}{2} + \frac{3}{2}e^{-t}} \rightarrow \boxed{i_L(t) = \begin{cases}\frac{5}{2} + \frac{3}{2}e^{-t} & \text{for } t>0 \\ 4 & \text{for } t<0\end{cases}}$$

*See top of page for graph