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EE103

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Midterm Solutions

Problem 1. (10 points). For what values of the scalar a are the following matrices positive definite?

$$A = \begin{bmatrix} I & aI \\ aI & I \end{bmatrix}, \qquad A = \begin{bmatrix} I & au \\ au^T & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a \\ 1 & a & 2 \end{bmatrix}.$$

I is the $n \times n$ identity matrix and $u = (1, 1, \ldots, 1)$, the n-vector with all its elements equal to one.

Solution. A matrix is positive definite if and only if it has a Cholesky factorization. The first matrix has a Cholesky factorization

$$A = \begin{bmatrix} I & 0 \\ aI & \sqrt{1-a^2}I \end{bmatrix} \begin{bmatrix} I & aI \\ 0 & \sqrt{1-a^2}I \end{bmatrix}$$

if and only if $|a| < 1$. The second matrix has a Cholesky factorization

$$A = \begin{bmatrix} I & 0 \\ au^T & \sqrt{1-a^2 n} \end{bmatrix} \begin{bmatrix} I & au \\ 0 & \sqrt{1-a^2 n} \end{bmatrix}$$

if and only if $|a| < 1/\sqrt{n}$. The third matrix has a Cholesky factorization

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \sqrt{a-1} & 0 \\ 1 & \sqrt{a-1} & \sqrt{2-a} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & \sqrt{a-1} & \sqrt{a-1} \\ 0 & 0 & \sqrt{2-a} \end{bmatrix}$$

if and only if $1 < a < 2$.

Problem 2 (10 points). Consider the linear equation

$$(A + \epsilon B)x = b,$$

where A and B are given $n \times n$-matrices, b is a given n-vector, and ϵ is a scalar parameter. We assume that A is nonsingular, and therefore $A+\epsilon B$ is nonsingular for sufficiently small ϵ. The solution of the equation is

$$x(\epsilon) = (A + \epsilon B)^{-1} b,$$

a complicated nonlinear function of ϵ. In order to find a simple approximation of $x(\epsilon)$, valid for small ϵ, we can expand $x(\epsilon) = (A + \epsilon B)^{-1} b$ in a series

$$x(\epsilon) = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots,$$

where x_0, x_1, x_2, x_3, ... are n-vectors, and then truncate the series after a few terms. To determine the coefficients x_i in the series, we examine the equation

$$(A + \epsilon B)(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots) = b.$$

Expanding the product on the lefthand side gives

$$Ax_0 + \epsilon(Ax_1 + Bx_0) + \epsilon^2(Ax_2 + Bx_1) + \epsilon^3(Ax_3 + Bx_2) + \cdots = b.$$

We see that if this holds for all ϵ in a neighborhood of zero, the coefficients x_i must satisfy

$$Ax_0 = b, \qquad Ax_1 + Bx_0 = 0, \qquad Ax_2 + Bx_1 = 0, \qquad Ax_3 + Bx_2 = 0, \qquad \ldots. \tag{1}$$

Describe an efficient method for computing the first $k+1$ coefficients $x_0, \ldots, x_k$ from (1). What is the complexity of your method (number of flops for large n, assuming $k \ll n$)? If you know several methods, give the most efficient one.

Solution. We need to solve $k + 1$ equations

$$Ax_0 = b, \qquad Ax_1 = -Bx_0, \qquad Ax_2 = -Bx_1, \qquad \ldots, \qquad Ax_k = -Bx_{k-1}.$$

This requires one LU factorization of A, k matrix-vector products Bx_i, and $k + 1$ forward and backward substitutions. The total cost is

$$(2/3)n^3 + 2kn^2 + 2(k+1)n^2 \approx (2/3)n^3.$$

Problem 3 (10 points). We define A as the $n \times n$ lower triangular matrix with diagonal elements 1, and elements -1 below the diagonal:

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & 1 & 0 \\ -1 & -1 & -1 & \cdots & -1 & 1 \end{bmatrix}.$$

1. What is A^{-1}?
2. Show that $\kappa(A) \geq 2^{n-2}$. This means that the matrix is ill-conditioned for large n.
3. Show with an example that small errors in the righthand side of $Ax = b$ can produce very large errors in x. Take $b = (0, 0, \ldots, 0, 1)$ (the n-vector with all its elements zero, except the last element, which is one), and find a nonzero Δb for which

$$\frac{\|\Delta x\|}{\|x\|} \geq 2^{n-2} \frac{\|\Delta b\|}{\|b\|},$$

where x is the solution of $Ax = b$ and $x + \Delta x$ is the solution of $A(x + \Delta x) = b + \Delta b$.

Solution.

1. We find A^{-1} by solving $AX = I$, column by column. This gives

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 2 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 4 & 2 & 1 & 1 & 0 & \cdots & 0 \\ 8 & 4 & 2 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{n-2} & 2^{n-3} & 2^{n-4} & 2^{n-5} & 2^{n-6} & \cdots & 1 \end{bmatrix}.$$

The diagonal elements are one; we have $(A^{-1})_{ij} = 2^{i-j-1}$ for $i > j$.

2. We bound the norm of A and A^{-1} using the inequalities

$$\|A\| \geq \frac{\|Ax\|}{\|x\|}, \qquad \|A^{-1}\| \geq \frac{\|A^{-1}y\|}{\|y\|},$$

which hold for all nonzero x and y. If we take $x = (0, \ldots, 0, 1)$ and $y = (1, 0, \ldots, 0)$ we get

$$\|A\| \geq \|Ax\| = 1, \qquad \|A^{-1}\| \geq \|A^{-1}y\| \geq 2^{n-2}.$$

Therefore

$$\kappa(A) = \|A\|\|A^{-1}\| \geq 2^{n-2}.$$

3. The solution of $Ax = b$ is $x = (0, 0, \ldots, 1)$. If we take $\Delta b = (1, 0, \ldots, 0)$, then $\Delta x = (1, 1, 2, 4, 8, \ldots, 2^{n-2})$. Therefore

$$\frac{\|\Delta x\|}{\|x\|} \geq 2^{n-2}, \qquad \frac{\|\Delta b\|}{\|b\|} = 1.$$

Problem 4 (10 points). Describe an efficient method for each of the following two problems, and give a complexity estimate (number of flops for large n).

1. Solve
$$DX + XD = B$$
where D is $n \times n$ and diagonal. The diagonal elements of D satisfy $d_{ii} + d_{jj} \neq 0$ for all i and j. The matrices D and B are given. The variable is the $n \times n$-matrix X.

2. Solve
$$LX + XL^T = B$$
where L is lower triangular. The diagonal elements of L satisfy $l_{ii} + l_{jj} \neq 0$ for all i and j. The matrices L and B are given. The variable is the $n \times n$-matrix X. (*Hint:* Solve for X column by column.)

If you know several methods, choose the fastest one (least number of flops for large n).

Solution.

1. The i, j-element of $DX + XD$ is $(d_{ii} + d_{jj})X_{ij}$. It is given that $d_{ii} + d_{jj} \neq 0$, so we can solve for X element by element:
$$X_{ij} = \frac{b_{ij}}{d_{ii} + d_{jj}}, \qquad i, j = 1, \ldots, n.$$
The cost is $2n^2$ flops.

2. Denote the columns of X by X_1, X_2, ..., X_n, and the columns of B by B_1, B_2, ..., B_n. If we write the equation in terms of the columns of X and B we get
$$\begin{aligned} LX_1 + l_{11}X_1 &= B_1 \\ LX_2 + l_{21}X_1 + l_{22}X_2 &= B_2 \\ LX_3 + l_{31}X_1 + l_{32}X_2 + l_{33}X_3 &= B_2 \\ &\vdots \\ LX_n + l_{n1}X_1 + l_{n2}X_2 + \cdots + l_{nn}X_n &= B_n. \end{aligned}$$
In other words,
$$\begin{aligned} (L + l_{11}I)X_1 &= B_1 \\ (L + l_{22}I)X_2 &= B_2 - l_{21}X_1 \\ (L + l_{33}I)X_3 &= B_2 - l_{31}X_3 - l_{32}X_2 \\ &\vdots \\ (L + l_{nn}I)X_n &= B_n - l_{n1}X_1 - l_{n2}X_2 - \cdots - l_{n,n-1}X_{n-1}. \end{aligned}$$

We can solve for X_1 first, then evaluate the righthand side of the second equation and solve for X_2, then plug in X_1 and X_2 in the third righthand side and solve for X_4, et cetera. The matrices on the left are upper triangular with nonzero diagonal elements (since we are given that $l_{ii} + l_{jj} \neq 0$ for all i and j), so each equation can be solved by forward substitution.

To estimate the cost we look at the equation for the kth column:

$$(L + l_{kk} I)X_k = B_k - l_{k1}X_1 - l_{k2}X_2 - \cdots - l_{k,k-1}X_{k-1}.$$

The vector on the right can be computed in $2(k-1)n$ operations. Adding l_{kk} to the diagonal elements of U takes n flops. Solving the equation costs n^2. The cost of step k is therefore $n^2 + (2k-1)n$. The total cost is

$$\begin{aligned} n^3 + n\sum_{k=1}^{n}(2k-1) &= n^3 + n(1 + 3 + 5 + \cdots + (2n-1)) \\ &= 2n^3. \end{aligned}$$