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EE103

5/6/03

Midterm Exam

DO NOT OPEN UNTIL EVERYONE IS READY TO START

- You have time until 9:15.
- Only **this booklet** should be on your desk. You do not need a calculator.
- Write your answers neatly and concisely in the space provided after each question. You can use the blank pages on the left as scratch paper. Provide enough detail to convince us that you derived, not guessed, your answers.

Your name: [redacted]

Your student ID#: [redacted]

Your left neighbor's name: [redacted]

Your right neighbor's name: [redacted]

Problem 1	15 /25
Problem 2	0 /25
Problem 3	16 /25
Problem 4	13 /25
Total	44 /100

Important formulas and definitions

Lecture 1. Vectors and matrices

- Relation between inner product and angle between two vectors: $x^T y = \|x\|\,\|y\| \cos \angle(x, y)$
- Flop counts for basic matrix and vector operations. Suppose $\alpha \in \mathbf{R}$, $x, y \in \mathbf{R}^n$, $A \in \mathbf{R}^{m \times n}$, $B \in \mathbf{R}^{n \times p}$.
  - Inner product $x^T y$: $2n - 1 \approx 2n$
  - Vector addition $x + y$: n
  - Scalar multiplication αx: n
  - Matrix-vector product Ax: $m(2n - 1) \approx 2mn$
  - Matrix-matrix product AB: $mp(2n - 1) \approx 2mpn$

Lecture 3. The solution of a set of linear equations

- Definition of matrix norm: $\|A\| = \max_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}$
- Properties of the matrix norm:

  $\|A\| = \max_{\|x\|=1} \|Ax\|$
  $\|\alpha A\| = |\alpha| \|A\|$ for $\alpha \in \mathbf{R}$
  $\|A\| \geq 0$ for all A; $\|A\| = 0$ iff $A = 0$
  $\|A + B\| \leq \|A\| + \|B\|$
  $\|Ax\| \leq \|A\|\,\|x\|$ for all $x \in \mathbf{R}^n$
  $\|AB\| \leq \|A\|\,\|B\|$
  $1/\|A^{-1}\| = \min_{x \neq 0}(\|Ax\|/\|x\|)$ if A is square and nonsingular
  $\|A\|\,\|A^{-1}\| \geq 1$ if A is square and nonsingular

- Definition of condition number: $\kappa(A) = \|A\|\,\|A^{-1}\|$
- Error bounds for $Ax = b$, $A(x + \Delta x) = b + \Delta b$:

$$\|\Delta x\| \leq \|A^{-1}\| \|\Delta b\|, \qquad \frac{\|\Delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\Delta b\|}{\|b\|}$$

Lecture 4. Solving linear equations

Cost of solving $Ax = b$ when $A \in \mathbf{R}^{n \times n}$ is upper or lower triangular: n^2 flops

Lecture 5. The LU factorization

Cost of LU factorization $A = PLU$: $(2/3)n^3$ flops if $A \in \mathbf{R}^{n \times n}$

Lecture 6. The Cholesky factorization

Cost of Cholesky factorization $A = LL^T$: $(1/3)n^3$ flops if $A \in \mathbf{R}^{n \times n}$

-10

Problem 1. (25 points)

We have encountered the matrix

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-2} & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-2} & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & t_{n-1} & t_{n-1}^2 & \cdots & t_{n-1}^{n-2} & t_{n-1}^{n-1} \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-2} & t_n^{n-1} \end{bmatrix}$$

in the context of polynomial interpolation. A matrix of this form is often badly conditioned. As an example, suppose

$$t_1 = 1, \quad t_2 = 2, \quad t_3 = 3, \quad \ldots, \quad t_{n-1} = n-1, \quad t_n = n.$$

Show that

$$\kappa(A) \geq n^{n-3/2}.$$

$\|A\| \|A^{-1}\| \geq 1$ if A is square & nonsingular

Answer for problem 1.

$\kappa(A) = \|A\| \|A^{-1}\|$ ✓

$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$ ✓ $\qquad \frac{1}{\|A^{-1}\|} = \min_{x \neq 0} \left(\frac{\|Ax\|}{\|x\|}\right)$ ✓

$\|A^{-1}\| = \frac{1}{\min_{x \neq 0}\left(\frac{\|Ax\|}{\|x\|}\right)}$ ✓

how?

$\|A\| \|A^{-1}\| = \frac{\max_{x \neq 0} \frac{\|Ax\|}{\|x\|}}{\min_{x \neq 0} \left(\frac{\|Ax\|}{\|x\|}\right)} = \frac{n^{n-1}}{n^{\frac{1}{2}}} = n^{n-\frac{3}{2}} = \geq n^{n-\frac{3}{2}}$

how?

need explanations!

-25

Problem 2. (25 points).

Suppose the matrix

$$A = \begin{bmatrix} I & B \\ C & I \end{bmatrix}$$

is nonsingular, where B is $m \times n$ and C is $n \times m$.
The cost of solving $Ax = b$ using the standard method for dense matrices is $(2/3)(n+m)^3$ flops, since A has size $(n+m) \times (n+m)$. Describe a more efficient method, assuming that n and m are large, $n > m$, and B and C are dense. If you know several methods, give the most efficient one.
Clearly state the different steps in your algorithm and give the cost (number of flops for large n, m) of each step. What is the total flop count (keeping only the leading terms)?

Answer for problem 2.

$+ m^2 + n^2$

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Problem 3. (25 points)

Suppose you are asked to write code for solving two sets of linear equations

$$Ax = b, \qquad (A + uv^T)y = b,$$

where A is $n \times n$ and given, and u, v, and b are given n-vectors. The variables are x and y. We assume that A and $A + uv^T$ are nonsingular.
The cost of solving the two systems from scratch is $\underline{(4/3)n^3}$ flops. Give a more efficient method, based on the expression

$$(A + uv^T)^{-1} = A^{-1} - \frac{1}{1 + \underbrace{v^T A^{-1} u}_{2n^2}} \overbrace{A^{-1} uv^T A^{-1}}^{2n} \rightarrow = 2n^2 + 2n + n^2$$

n^2

(You do not have to derive this formula.)
Clearly state the different steps in your algorithm and give the flop count of each step (for large n). What is the total flop count (for large n)?

Answer for problem 3.

$\frac{2}{3}n^3 +$

Ax

$$A^{-1} - \frac{1}{1 + \underbrace{v^T A^{-1} u}_{+2n^2}} \leftarrow 1$$

n^2

$+2n$

$+1$

$uv^T = v^T u$

$$\Big[\underbrace{A^{-1} u}_{2n^2} \; \underbrace{v^T \quad A^{-1}}_{2n^2}\Big] y = b$$

$2n$

$= 7n \quad 6n + 2$

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Problem 4. (25 points)

A matrix A is *tridiagonal* if $a_{ij} = 0$ for $|i - j| > 1$, *i.e.*, A has the form

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2,n-2} & a_{n-2,n-1} & 0 \\ 0 & 0 & 0 & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & a_{n,n-1} & a_{nn} \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 2 & 8 & -2 & 0 & 0 \\ 0 & -2 & 3 & -1 & 0 \\ 0 & 0 & -1 & 8 & -2 \\ 0 & 0 & 0 & -2 & 3 \end{bmatrix}$$

is an example of a 5×5 symmetric tridiagonal matrix.
What is the cost of computing the Cholesky factorization of a general tridiagonal positive definite matrix of size $n \times n$? Count square roots as one flop and keep only the leading term in the total number of flops. Explain your answer.

Answer for problem 4.

$$\begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 2 & 8 & -2 & 0 & 0 \\ 0 & -2 & 3 & -1 & 0 \\ 0 & 0 & -1 & 8 & -2 \\ 0 & 0 & 0 & -2 & 3 \end{bmatrix} = \overset{L}{\begin{bmatrix} l_{11} & 0 & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} & 0 \\ l_{51} & l_{52} & l_{53} & l_{54} & l_{55} \end{bmatrix}} \overset{L^T}{\begin{bmatrix} l_{11} & l_{21} & l_{31} & l_{41} & l_{51} \\ 0 & l_{22} & l_{32} & l_{42} & l_{52} \\ 0 & 0 & l_{33} & l_{43} & l_{53} \\ 0 & 0 & 0 & l_{44} & l_{54} \\ 0 & 0 & 0 & 0 & l_{55} \end{bmatrix}}$$

For first column of L takes $l_{11} \cdot l_{11} = 2$
2n flops

$l_{21}^2 + l_{22}^2 = 8$ ✓
2 1 2
[illegible]

?

n flops

$9(n-1)$ flops for 2nd column of

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 3$
2 1 2 1 2
1 substract
1 square root

$10(n-2)$. 3rd column of L