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EE103

Midterm solutions

Problem 1 (10 points). A square matrix A is called *normal* if $AA^T = A^TA$. Show that if A is normal and nonsingular, then the matrix $Q = A^{-1}A^T$ is orthogonal.

Solution. We show that $Q^TQ = I$:

$$\begin{aligned} Q^TQ &= (A^{-1}A^T)^T(A^{-1}A^T) \\ &= AA^{-T}A^{-1}A^T \\ &= A(AA^T)^{-1}A^T \\ &= A(A^TA)^{-1}A^T \\ &= AA^{-1}A^{-T}A^T \\ &= I. \end{aligned}$$

On line 2 we use $(BC)^T = C^TB^T$. On lines 3 and 5 we use the property that $(BC)^{-1} = C^{-1}B^{-1}$ if B and C are invertible. On line 4 we use the definition of normal matrix.

Problem 2 (10 points). Let $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, $c = (c_1, c_2, c_3)$, $d = (d_1, d_2, d_3)$ be four given points in $\mathbf{R}^3$.

The points do not lie in one plane. Algebraically, this can be expressed by saying that the vectors $b-a$, $c-a$, $d-a$ are linearly independent, *i.e.*, $y_1(b-a)+y_2(c-a)+y_3(d-a) = 0$ only if $y_1 = y_2 = y_3 = 0$.

Suppose we are given the distances of a point $x = (x_1, x_2, x_3)$ to the four points:

$$\|x-a\| = r_a, \qquad \|x-b\| = r_b, \qquad \|x-c\| = r_c, \qquad \|x-d\| = r_d.$$

Write a set of linear equations $Ax = f$, with A nonsingular, from which the coordinates x_1, x_2, x_3 can be computed. Explain why the matrix A is nonsingular.

Solution. Squaring the norms gives four nonlinear equations

$$\begin{aligned} \|x\|^2 - 2a^Tx + \|a\|^2 &= r_a^2 \\ \|x\|^2 - 2b^Tx + \|b\|^2 &= r_b^2 \\ \|x\|^2 - 2c^Tx + \|c\|^2 &= r_c^2 \\ \|x\|^2 - 2d^Tx + \|d\|^2 &= r_d^2. \end{aligned}$$

Subtracting the first equation from the three others gives

$$\begin{aligned} 2(a-b)^Tx &= r_b^2 - r_a^2 + \|a\|^2 - \|b\|^2 \\ 2(a-c)^Tx &= r_c^2 - r_a^2 + \|a\|^2 - \|c\|^2 \\ 2(a-d)^Tx &= r_d^2 - r_a^2 + \|a\|^2 - \|d\|^2. \end{aligned}$$

In matrix form,

$$\begin{bmatrix} a_1 - b_1 & a_2 - b_2 & a_3 - b_3 \\ a_1 - c_1 & a_2 - c_2 & a_3 - c_3 \\ a_1 - d_1 & a_2 - d_2 & a_3 - d_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} r_b^2 - r_a^2 + \|a\|^2 - \|b\|^2 \\ r_c^2 - r_a^2 + \|a\|^2 - \|c\|^2 \\ r_d^2 - r_a^2 + \|a\|^2 - \|d\|^2 \end{bmatrix}.$$

The matrix is nonsingular because its rows are linearly independent. (Equivalently, A^T has a zero nullspace, so it is nonsingular and therefore A is nonsingular.)

Problem 3 (15 points). Let A be a nonsingular $n \times n$-matrix and let u, v be two n-vectors that satisfy $v^T A^{-1} u \neq 1$.

1. Show that

$$\begin{bmatrix} A & u \\ v^T & 1 \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{1 - v^T A^{-1} u} \begin{bmatrix} A^{-1} u \\ -1 \end{bmatrix} \begin{bmatrix} v^T A^{-1} & -1 \end{bmatrix}.$$

2. Describe an efficient method for solving the two equations

$$Ax = b, \qquad \begin{bmatrix} A & u \\ v^T & 1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}.$$

   The variables are the n-vectors x and y, and the scalar z.

   Describe in detail the different steps in your algorihm and give a flop count of each step. If you know several methods, choose the most efficient one (least number of flops for large n).

Solution.

1. We show that the product of the two matrices is the identity:

$$\begin{aligned} &\begin{bmatrix} A & u \\ v^T & 1 \end{bmatrix} \left(\begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{1 - v^T A^{-1} u} \begin{bmatrix} A^{-1} u \\ -1 \end{bmatrix} \begin{bmatrix} v^T A^{-1} & -1 \end{bmatrix} \right) \\ &= \begin{bmatrix} I & 0 \\ v^T A^{-1} & 0 \end{bmatrix} + \frac{1}{1 - v^T A^{-1} u} \begin{bmatrix} 0 \\ -1 + v^T A^{-1} u \end{bmatrix} \begin{bmatrix} v^T A^{-1} & -1 \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ v^T A^{-1} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} v^T A^{-1} & -1 \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ v^T A^{-1} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -v^T A^{-1} & 1 \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

2. We are asked to compute $x = A^{-1}b$ and

$$
\begin{aligned}
\begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} A & u \\ v^T & 1 \end{bmatrix}^{-1} \begin{bmatrix} b \\ c \end{bmatrix} \\
&= \left(\begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{1 - v^T A^{-1} u} \begin{bmatrix} A^{-1}u \\ -1 \end{bmatrix} \begin{bmatrix} v^T A^{-1} & -1 \end{bmatrix} \right) \begin{bmatrix} b \\ c \end{bmatrix} \\
&= \begin{bmatrix} A^{-1}b \\ 0 \end{bmatrix} + \frac{v^T A^{-1} b - c}{1 - v^T A^{-1} u} \begin{bmatrix} A^{-1}u \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} x \\ 0 \end{bmatrix} + \frac{v^T x - c}{1 - v^T A^{-1} u} \begin{bmatrix} A^{-1}u \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} x \\ 0 \end{bmatrix} + \frac{v^T x - c}{1 - v^T w} \begin{bmatrix} w \\ -1 \end{bmatrix}
\end{aligned}
$$

if we define $w = A^{-1}u$. We can compute x, y, z as follows.

- LU factorization $A = PLU$ ($(2/3)n^3$ flops).
- Solve $PLUx = b$ by forward and backward substitution ($2n^2$ flops).
  - Solve $Px_1 = b$ by applying a permutation to b: $x_1 = P^T b$ (0 flops).
  - Solve $Lx_2 = x_1$ by forward substitution (n^2 flops).
  - Solve $Ux = x_2$ by back substitution (n^2 flops).
- Solve $PLUw = u$ in the same way ($2n^2$ flops).
- Compute $z = (v^T x - c)/(v^T w - 1)$ ($4n + 1$ flops).
- Compute $y = x - zw$ ($2n$ flops).

The total (for large n) is $(2/3)n^3$ flops.

Problem 4 (15 points). Let A be a positive definite 5×5-matrix with the nonzero pattern

$$
A = \begin{bmatrix} \bullet & & \bullet & \bullet & \\ & \bullet & & \bullet & \\ \bullet & & \bullet & & \bullet \\ \bullet & \bullet & & \bullet & \\ & & \bullet & & \bullet \end{bmatrix}.
$$

The dots indicate the positions of the nonzero elements; all the other elements are zero.

The Cholesky factor L of A has one of the following lower-triangular nonzero patterns. Which one is correct? For each L, explain why L is or is not the Cholesky factor of A.

(a) $$L = \begin{bmatrix} \bullet & & & & \\ & \bullet & & & \\ \bullet & & \bullet & & \\ & \bullet & & \bullet & \\ \bullet & & \bullet & & \bullet \end{bmatrix}$$

(b) $$L = \begin{bmatrix} \bullet & & & & \\ & \bullet & & & \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & & \bullet & \\ & & \bullet & & \bullet \end{bmatrix}$$

(c) $L = \begin{bmatrix} \bullet & & & & \\ & \bullet & & & \\ \bullet & & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & \\ & \bullet & & \bullet & \bullet \end{bmatrix}$ (d) $L = \begin{bmatrix} \bullet & & & & \\ & \bullet & & & \\ \bullet & & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & \\ & & \bullet & \bullet & \bullet \end{bmatrix}$

Solution. Recall the general algorithm for a Cholesky factorization

$$\begin{bmatrix} a_{11} & A_{21}^T \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} l_{11} & L_{21}^T \\ 0 & L_{22}^T \end{bmatrix}.$$

- Determine the first column: $l_{11} = \sqrt{a_{11}}$, $L_{21} = (1/l_{11})A_{21}$.
- Factor $A_{22} - L_{21}L_{21}^T = L_{22}L_{22}^T$.

From the first step we see that the first column of L must have the same nonzero pattern as the first column of A. This already rules out the matrix (a) because of the nonzero in position $4, 1$ of L.

After the first step of the factorization, the matrix $A_{22} - L_{21}L_{21}^T$ has the following pattern:

$$A_{22} - L_{21}L_{21}^T = \begin{bmatrix} \bullet & & \bullet & \\ & \bullet & & \bullet \\ \bullet & & \bullet & \\ & \bullet & & \bullet \end{bmatrix} - \begin{bmatrix} \\ \bullet \\ \bullet \\ \\ \end{bmatrix} \begin{bmatrix} \\ \bullet \\ \bullet \\ \\ \end{bmatrix}^T = \begin{bmatrix} \bullet & & \bullet & \\ & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ & \bullet & & \bullet \end{bmatrix}$$

The first column of the Cholesky factor L_{22} of this matrix must have zeros in positions 2 and 4. In other words, elements 3 and 5 of the second column of L must be zero. This rules out the matrices (b) and (c).

Continuing the same reasoning, we see that in step 3 we factor a matrix with nonzero pattern

$$\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \bullet \end{bmatrix} - \begin{bmatrix} \\ \bullet \\ \\ \end{bmatrix} \begin{bmatrix} \\ \bullet \\ \\ \end{bmatrix}^T = \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \bullet \end{bmatrix},$$

and in the step 4 a matrix

$$\begin{bmatrix} \bullet & \\ & \bullet \end{bmatrix} - \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} \bullet \\ \bullet \end{bmatrix}^T = \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix}.$$

We see that all the elements below the diagonal in the third and fourth columns of L are nonzero. This is consistent with the pattern (d).