L. Vandenberghe
EE103

12/12/00

Final Exam

DO NOT OPEN UNTIL EVERYONE IS READY TO START

- You have time until 2:30.
- Only **this booklet** should be on your desk. You do not need a calculator.
- Write your answers neatly and concisely in the space provided after each question. You can use the blank pages on the left as scratch paper. Provide enough detail to convince us that you derived, not guessed, your answers.
- The list of formulas is identical to the list posted on the website, except for 'Lecture 13', where we corrected a typo and added two formulas.

Your name: ____________________

Your student ID#: ____________________

Your left neighbor's name: ____________________

Your right neighbor's name: ____________________

Problem 1	/20
Problem 2	/20
Problem 3	/20
Problem 4	/20
Problem 5	/20
Problem 6	/20
Total	/120

Important formulas and definitions

Lecture 2. Accuracy of numerical algorithms

- Floating-point numbers with base 2

$$\pm(.d_1d_2\ldots d_n)_2 \cdot 2^e = \pm(d_1 2^{-1} + d_2 2^{-2} + \cdots + d_n 2^{-n}) \cdot 2^e$$

  with $d_1 = 1$, $d_i \in \{0, 1\}$
- Machine precision: $\epsilon_M = 2^{-n}$
- IEEE double precision arithmetic: $-1021 \le e \le 1024$, $n = 53$, $\epsilon_M \approx 1.11 \cdot 10^{-16}$

Lecture 3. Vectors and matrices

- Geometric interpretation of inner product: $x^T y = \|x\|\|y\| \cos \angle(x, y)$
- Number of flops for basic matrix and vector operations:
  - inner product $x^T y$ where $x, y \in \mathbf{R}^n$: $2n$ flops
  - vector addition $x + y$, scalar multiplication αx where $x, y \in \mathbf{R}^n$, $\alpha \in \mathbf{R}$: n flops
  - matrix-vector multiplication Ax where $A \in \mathbf{R}^{m\times n}$: $2mn$ flops
  - matrix-matrix multiplication AB where $A \in \mathbf{R}^{m\times p}$, $B \in \mathbf{R}^{p\times n}$: $2mnp$ flops

Lecture 5. The solution of a set of linear equations

- Definition of matrix norm: $\|A\| = \max_{x\neq 0} \dfrac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$
- Properties of the matrix norm:

  $\|\alpha A\| = |\alpha|\|A\|$ for $\alpha \in \mathbf{R}$
  $\|A\| \ge 0$ for all A; $\|A\| = 0$ iff $A = 0$
  $\|A + B\| \le \|A\| + \|B\|$
  $\|Ax\| \le \|A\|\|x\|$ for all $x \in \mathbf{R}^n$
  $\|AB\| \le \|A\|\|B\|$
  $1/\|A^{-1}\| = \min_{x\neq 0}(\|Ax\|/\|x\|)$ if A is square and nonsingular
  $\|A\|\|A^{-1}\| \ge 1$ if A is square and nonsingular

- Definition of condition number: $\kappa(A) = \|A\|\|A^{-1}\|$
- Error bounds for $Ax = b$, $A(x + \Delta x) = b + \Delta b$:

$$\|\Delta x\| \le \|A^{-1}\|\|\Delta b\|, \qquad \frac{\|\Delta x\|}{\|x\|} \le \kappa(A)\frac{\|\Delta b\|}{\|b\|}$$

Lecture 6. Solving sets of linear equations

- cost of solving $Ax = b$ when $A \in \mathbf{R}^{n \times n}$ is upper or lower triangular: n^2 flops
- LU factorization with partial pivoting: $A = PLU$ (P a permutation matrix, L unit lower triangular, U upper triangular). Cost: $2n^3/3$ flops if $A \in \mathbf{R}^{n \times n}$

Lecture 7. Positive definite sets of linear equations

- Cholesky factorization: $A = LL^T$ (L lower triangular with positive diagonal elements). Cost: $n^3/3$ flops if $A \in \mathbf{R}^{n \times n}$

Lecture 9. The solution of a least-squares problem

- Solution that minimizes $\|Ax - b\|$ if $A \in \mathbf{R}^{m \times n}$ has rank n: $x = (A^T A)^{-1} A^T b$

Lecture 10. Solving least-squares problems

- QR-factorization: $A = QR$ (Q orthogonal, R upper triangular with positive diagonal elements). Cost: $2mn^2$ flops if $A \in \mathbf{R}^{m \times n}$

Lecture 11. Underdetermined sets of linear equations

- Least-norm solution of $Ax = b$ if $A \in \mathbf{R}^{m \times n}$ has rank m: $x = A^T (AA^T)^{-1} b$

Lecture 12. Nonlinear equations

- Newton iteration for solving a set of nonlinear equations $f(x) = 0$ where $f : \mathbf{R}^n \to \mathbf{R}^n$: $x^+ = x - Df(x)^{-1} f(x)$, where $Df(x) \in \mathbf{R}^{n \times n}$ with $(Df(x))_{ij} = \partial f_i(x)/\partial x_j$.

Lecture 13. Unconstrained minimization

- Newton iteration for minimizing a function $g : \mathbf{R}^n \to \mathbf{R}$: $x^+ = x - \nabla^2 g(x)^{-1} \nabla g(x)$, where $\nabla g(x) \in \mathbf{R}^n$ with $\nabla g(x)_i = \partial g(x)/\partial x_i$, and $\nabla^2 g(x) \in \mathbf{R}^{n \times n}$ with $(\nabla^2 g(x))_{ij} = \partial^2 g(x)/\partial x_i \partial x_j$.
- if $g(x) = \sum_{i=1}^m r_i(x)^2$, where $r_i : \mathbf{R}^n \to \mathbf{R}$, then

$$\nabla g(x) = 2 \sum_{i=1}^m r_i(x) \nabla r_i(x), \quad \nabla^2 g(x) = 2 \sum_{i=1}^m (r_i(x) \nabla^2 r_i(x) + \nabla r_i(x) \nabla r_i(x)^T).$$

Problem 1. (20 points)

If x and y are two positive numbers with $x \neq y$, then

$$\frac{x+y}{2} - \sqrt{xy} > 0.$$

(The arithmetic mean $(x+y)/2$ is always greater than the geometric mean $\sqrt{xy}$.) The following Matlab code evaluates the left hand side of the inequality at $x = 1$ and $y = 1.01$, rounding the result of the square root to 4 significant digits:

```
>>  x=1;  y=1.01;
>>  (x+y)/2 - chop(sqrt(x*y),4)
ans =
    -2.2204e-016
```

The result is clearly wrong because it should be positive. (The correct answer is $1.24 \cdot 10^{-5}$.)

1. Explain why the result is wrong. (You do not have to explain the numerical value of the result.)
2. Give a more stable method for evaluating $(x+y)/2 - \sqrt{xy}$ when $x \approx y$. (You should assume that square roots are evaluated with 4 correct significant digits, and that multiplications, divisions, additions, and subtractions are exact.)

Answer for problem 1.

Problem 2. (20 points)

A matrix C is defined as

$$C = Auv^T B$$

where $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times n}$, $u \in \mathbf{R}^n$, and $v \in \mathbf{R}^n$. The product on the right hand side can be evaluated in many different ways, *e.g.*, as $A(u(v^T B))$ or as $A((uv^T)B)$, etc. What is the fastest method (*i.e.*, requiring the least number of flops) when n is large? Explain your answer.

Answer for problem 2.

Problem 3. (20 points)

You are given a nonsingular matrix A, and you would like to obtain an idea of the condition number $\kappa(A)$ without calculating the exact value of $\kappa(A)$. To estimate the condition number you evaluate the matrix-vector product Ax for a few different values of x, and calculate the norm of x and the norm of Ax. The following table shows the results for four vectors $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, $x^{(4)}$.

	$\|x^{(i)}\|$	$\|Ax^{(i)}\|$
$i = 1$	1	100
$i = 2$	100	1
$i = 3$	10^3	10^4
$i = 4$	10^{-3}	10^2

What are the best (*i.e.*, largest) lower bounds on $\|A\|$, $\|A^{-1}\|$ and $\kappa(A)$ that you can derive based on this information?

Answer for problem 3.

Problem 4. (20 points)

Consider the set of $p+q$ linear equations in $p+q$ variables

$$\begin{bmatrix} I & A \\ A^T & -I \end{bmatrix} \begin{bmatrix} \bar{y} \\ \bar{x} \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}. \tag{1}$$

$A \in \mathbf{R}^{p\times q}$, $b \in \mathbf{R}^p$, and $c \in \mathbf{R}^q$ are given. The variables are $\bar{x} \in \mathbf{R}^q$ and $\bar{y} \in \mathbf{R}^p$.

1. Show that the coefficient matrix
$$\begin{bmatrix} I & A \\ A^T & -I \end{bmatrix}$$
is nonsingular, regardless of the rank and the dimensions of A. *Hint.* Show that the matrix $I + A^TA$ is nonsingular.

2. We can conclude from part 1 that the solution $\bar{x}$, $\bar{y}$ of (1) is unique. Show that $\bar{x}$ minimizes
$$\|Ax - b\|^2 + \|x + c\|^2.$$

Answer for problem 4.

Problem 5. (20 points)

Consider the underdetermined set of linear equations

$$Ax + By = b \tag{2}$$

where $b \in \mathbf{R}^p$, $A \in \mathbf{R}^{p\times p}$, and $B \in \mathbf{R}^{p\times q}$ are given. The variables are $x \in \mathbf{R}^p$ and $y \in \mathbf{R}^q$. We assume that $q < p$, that A is nonsingular, and that B is full rank (*i.e.*, $\mathbf{Rank}\, B = q$). The equations are underdetermined, so there are infinitely many solutions. For example, we can pick any y, and solve the set of linear equations $Ax = b - By$ to find x.

Below we define four solutions that minimize some measure of the magnitude of x, or y, or both. *For each of these solutions, describe the factorizations (QR, Cholesky, or LU with partial pivoting) that you would use to calculate x and y.* Clearly specify the matrices that you factorize, and the type of factorization. If you know several methods, you should give the most efficient one (least number of flops for large p, q).

1. The solution x, y with the smallest value of $\|x\|^2 + \|y\|^2 = \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2$.
2. The solution x, y with the smallest value of $\|x\|^2 + 2\|y\|^2$.
3. The solution x, y with the smallest value of $\|y\|^2$.
4. The solution x, y with the smallest value of $\|x\|^2$.

Answer for problem 5.

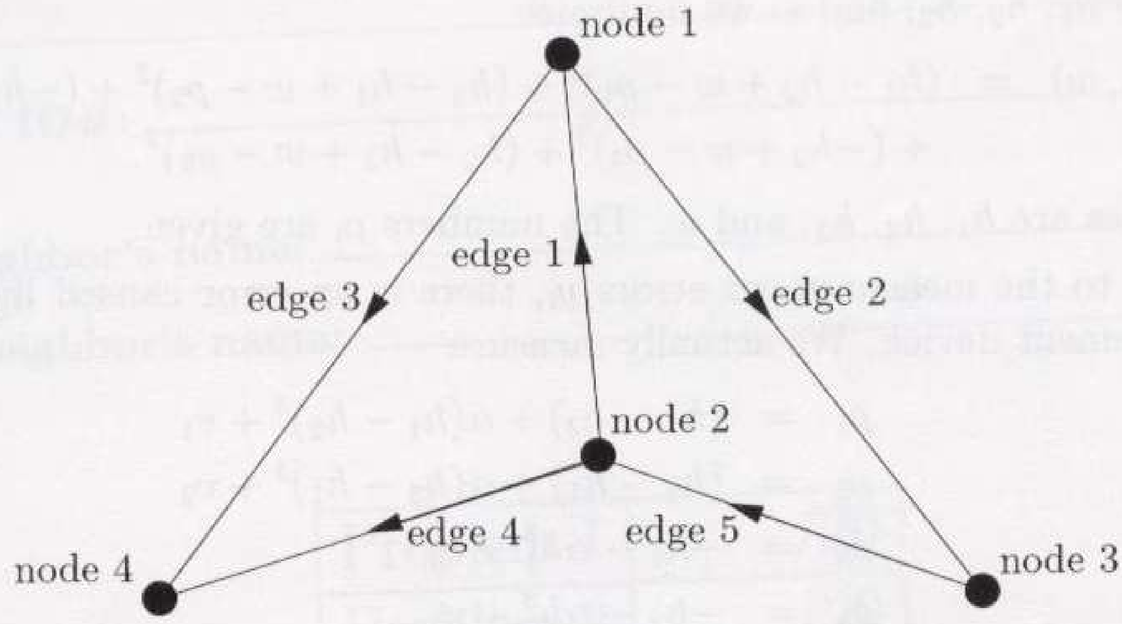
Problem 6. (20 points)

Below we define three functions g that we want to minimize (see equations (3), (4), (5)). For each of these three functions, explain which of the following three methods you would use to minimize g: linear least-squares via QR factorization, Newton's method, or the Gauss-Newton method.

- If your answer is linear least-squares, you should express the problem in the standard form of minimizing $\|Ax - b\|^2$, and give the matrix A, the vector of variables x, and the right hand side b.
- If your answer is Newton's method, you should describe how you construct the Hessian and gradient of g, and how you would choose the starting point.
- If your answer is Gauss-Newton method, you should describe how you construct the linear least-squares problems that you have to solve at each iteration, and how you would choose the starting point.

Note: The description of the problem is long, but all the information you need is summarized in the definitions of g (*i.e.*, the equations (3), (4), (5)). The rest of the discussion is background and can be read quickly.

The figure shows a *leveling network* used to determine the vertical height (or elevation) of three landmarks.



The nodes in the graph represent four points on a map. The variables in the problem are the heights h_1, h_2, h_3 of the first three points with respect to the fourth, which will be used as reference (in other words we take $h_4 = 0$). Each of the five edges in the graph represents a measurement of the height difference of its two end points. For example, edge 1 from node 2 to node 1 indicates a measurement of $h_1 - h_2$. We will denote the measured values of the five height differences as ρ_1, ρ_2, ρ_3, ρ_4, ρ_5. Our goal is to determine h_1, h_2, and h_3 from the five measurements. We distinguish three situations.

1. There is a small error in each measurement. The measured values are

$$\begin{aligned} \rho_1 &= h_1 - h_2 + v_1 \\ \rho_2 &= h_3 - h_1 + v_2 \\ \rho_3 &= -h_1 + v_3 \\ \rho_4 &= -h_2 + v_4 \\ \rho_5 &= h_2 - h_3 + v_5 \end{aligned}$$

where $v_1, \ldots, v_5$ are small and unknown measurement errors. To estimate the heights, we minimize the function

$$\begin{aligned} g(h_1, h_2, h_3) &= (h_1 - h_2 - \rho_1)^2 + (h_3 - h_1 - \rho_2)^2 + (-h_1 - \rho_3)^2 \\ &\quad + (-h_2 - \rho_4)^2 + (h_2 - h_3 - \rho_5)^2. \end{aligned} \tag{3}$$

The variables are h_1, h_2, h_3. The numbers ρ_i are given.

2. In addition to the measurement errors v_i, there is a systematic error, or offset, w in the measurement device. We actually measure

$$\begin{aligned} \rho_1 &= h_1 - h_2 + v_1 + w \\ \rho_2 &= h_3 - h_1 + v_2 + w \\ \rho_3 &= -h_1 + v_3 + w \\ \rho_4 &= -h_2 + v_4 + w \\ \rho_5 &= h_2 - h_3 + v_5 + w. \end{aligned}$$

To estimate h_1, h_2, h_3, and w we minimize

$$\begin{aligned} g(h_1, h_2, w) &= (h_1 - h_2 + w - \rho_1)^2 + (h_3 - h_1 + w - \rho_2)^2 + (-h_1 + w - \rho_3)^2 \\ &\quad + (-h_2 + w - \rho_4)^2 + (h_2 - h_3 + w - \rho_5)^2. \end{aligned} \tag{4}$$

The variables are h_1, h_2, h_3, and w. The numbers ρ_i are given.

3. In addition to the measurement errors v_i, there is an error caused by nonlinearity of the measurement device. We actually measure

$$\begin{aligned} \rho_1 &= (h_1 - h_2) + \alpha(h_1 - h_2)^3 + v_1 \\ \rho_2 &= (h_3 - h_1) + \alpha(h_3 - h_1)^3 + v_2 \\ \rho_3 &= -h_1 - \alpha h_1^3 + v_3 \\ \rho_4 &= -h_2 - \alpha h_2^3 + v_4 \\ \rho_5 &= (h_2 - h_3) + \alpha(h_2 - h_3)^3 + v_5. \end{aligned}$$

where $\alpha \in \mathbf{R}$ is small and given. To estimate h_1, h_2, h_3, we minimize

$$\begin{aligned} g(h_1, h_2, h_3) &= \left((h_1 - h_2) + \alpha(h_1 - h_2)^3 - \rho_1\right)^2 + \left((h_3 - h_1) + \alpha(h_3 - h_1)^3 - \rho_2\right)^2 \\ &\quad + \left(-h_1 - \alpha h_1^3 - \rho_3\right)^2 + \left(-h_2 - \alpha h_2^3 - \rho_4\right)^2 \\ &\quad + \left((h_2 - h_3) + \alpha(h_2 - h_3)^3 - \rho_5\right)^2. \end{aligned} \tag{5}$$

The variables are h_1, h_2, and h_3. The numbers ρ_i and α are given. You know that α is small.