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Average 59

Std Dev 17

EE 103 – MIDTERM EXAMINATION
Spring 2014

Instructions:

(a) The exam is closed-book (except for one page of notes) and will last 90 minutes.

(b) You may use a calculator to do arithmetic but use of cell phones or any other electronic devices is prohibited.

(c) Notation will conform as closely as possible to that used in the lectures.

(d) Do all 5 problems. Each problem is worth 20 points. Some partial credit may be assigned if warranted.

(e) Please put your full name, student number, and discussion section on the paper you turn in. Failure to include any of these will result in a loss of 5 points.

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1. Consider the solution of the $n \times n$ linear system

u v^T

$$(I + uv^T)x = b$$

where u, v, and b are given vectors in $\mathbb{R}^n$.

(a) Under what conditions on u and v does a solution x exist for all b and, when it exists, what is the solution?
Hint: Determine $\det(I + uv^T)$.

\# of multiplications

(b) Give an approximate operation count for getting the solution to this system.

a) $\det(I + uv^T) = 1 + v^T u$

(since $\det(I - xy^T) = 1 - y^T x$

$\boxed{\text{Condition: } v^T u \neq -1}$

By Sherman-Morrison-Woodbury

$A = I \quad B = u \quad C = v^T \quad D = 1$

If condition met

+9

$$\boxed{\begin{aligned} x &= (I + uv^T)^{-1} b \\ &= (I - u(1 + v^T u)^{-1} v^T) b \end{aligned}}$$

$(I + uv^T)^{-1} = (I)^{-1} - (I)^{-1} u (1^{-1} + v^T (I^{-1}) u)^{-1} v^T (I)^{-1}$

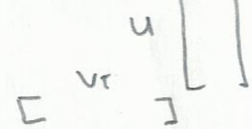
$= I - Iu(1 + v^T u)^{-1} v^T (I)^{-1}$

$= I - Iu(1 + v^T u)^{-1} v^T I$

$= I - u(1 + v^T u)^{-1} v^T I$ +4

$\quad\quad\quad n\times 1 \quad 1\times 1 \quad 1\times n$

$\Longrightarrow$



b)

$$x = (I - u(1+v^T u) v^T) b$$

$n \times n \quad n\times 1 \quad 1\times 1 \quad 1\times n \quad n\times 1$

n operations multiply v^T and u +1

n operations multiply u and $(1+v^T u)$ +1

n^2 operations multiply $u(1+v^T u)$ and v^T +1

n^2 operations multiply $(I-u(1+v^T u)v^T)$ and b

Approximate operation count of multiplications

n^2+n^2+n+n

$$\boxed{2n^2+2n}$$

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2. Let $A \in \mathbb{R}_n^{n \times n}$ have an LU factorization $A = LU$. Determine the LU factorization of the $2n \times 2n$ block matrix $M = \begin{bmatrix} A & A \\ A & cA \end{bmatrix}$ where c is a parameter. Under what conditions on c is M invertible?

$A = LU$

$M = LU$

$$\begin{bmatrix} A & A \\ A & cA \end{bmatrix} = \boxed{\overset{L}{\begin{bmatrix} L & 0 \\ L & -L \end{bmatrix}} \overset{U}{\begin{bmatrix} U & U \\ 0 & -(U - cU) \end{bmatrix}}}$$

+13

See work $\Longrightarrow$

$\boxed{M \text{ is invertible iff. } c \neq 1}$

+5

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3. Let $A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ where α is a nonnegative (possibly large) scalar.

$\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix}$

(a) Compute $\kappa_\infty(A)$.

(b) It is a fact that if $A \in \mathbb{R}_n^{n \times n}$, then $A + E$ is nonsingular if $\|A^{-1}\| \, \|E\| < 1$ or, equivalently, if $\frac{\|E\|}{\|A\|} < \frac{1}{\kappa(A)}$. Stated differently, if $A + E$ is singular, then $\frac{\|E\|}{\|A\|} \geq \frac{1}{\kappa(A)}$, thereby giving a lower bound on $\kappa(A)$. In fact, it can be shown that

$$\frac{1}{\kappa(A)} = \min\left\{\frac{\|E\|}{\|A\|} : A + E \text{ is singular}\right\}.$$

Therefore, $\kappa(A)$ is large if and only if there exists a singular matrix $A+E$ nearby in the sense that $\frac{\|E\|}{\|A\|}$ is small. For the specific matrix A in this problem, verify the formula above using the ∞-norm by considering $E = \begin{bmatrix} -\frac{1}{1+\alpha} & 0 \\ \frac{1}{1+\alpha} & 0 \end{bmatrix}$.

a) $A^{-1} = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix}$

$\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$

$= (1+\alpha)(1+\alpha)$

$\boxed{\kappa_\infty(A) = (1+\alpha)^2}$ +5

b) verify $\frac{1}{\kappa(A)} = \min\left\{ \frac{\|E\|}{\|A\|} : A+E \text{ singular} \right\}$

check?

we know $\kappa_\infty(A) = (1+\alpha)^2$ and $\|A\|_\infty = (1+\alpha)$

$\|E\|_\infty = \frac{1}{(1+\alpha)}$, so $\frac{1}{(1+\alpha)^2} = \min\left\{ \frac{\frac{1}{(1+\alpha)}}{(1+\alpha)} \right\} = \min\left\{ \frac{1}{(1+\alpha)^2} \right\}$

$= \frac{1}{(1+\alpha)^2}$ +5

$\square$

$$\begin{bmatrix} A & A \\ 0 & A - cA \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}\begin{bmatrix} A & A \\ A & A - A + cA \end{bmatrix}$$

$$\begin{bmatrix} U & U \\ 0 & U - cU \end{bmatrix}$$

$$\begin{bmatrix} L & 0 \\ L & -L \end{bmatrix}\begin{bmatrix} LU & LU \\ LU & LU - L(U - cU) \end{bmatrix}$$

$x_1 = 1$ $\quad y_1 = -2$
$x_2 = 2$ $\quad y_2 = -1$
$x_3 = 3$ $\quad y_3 = 3$

4. (a) Write the *Lagrangian form* of the polynomial that interpolates the points $(1, -2)$, $(2, -1)$, and $(3, 3)$. (Note that there is no need to simplify the polynomial.)
   (b) Write the *power form* of the polynomial above, i.e., simplify the polynomial.
   (c) Where does the polynomial that interpolates this data cross the x-axis?

a) $$\boxed{P(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)}(-2) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(-1) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(3)}$$

b) $$= \frac{x^2-5x+6}{-1(-2)}(-2) + \frac{x^2-4x+3}{(1)(-1)}(-1) + \frac{x^2-3x+2}{2(1)}(3)$$

$$= -(x^2-5x+6) + x^2-4x+3 + \frac{3}{2}(x^2-3x+2)$$

$$= \frac{3}{2}x^2 + x - \frac{9}{2}x - 3 + 3$$

$$= \boxed{\frac{3}{2}x^2 - \frac{7}{2}x}$$

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c) $x\left(\frac{3}{2}x - \frac{7}{2}\right) = 0$ $\quad 3x = 7$

$\boxed{\text{Crosses at } x = 0,\ x = 7/3}$ ✓

$f(1) = 1 - 5 = -4$
$f'(1) = 2(1) = 2$

5. Suppose it is desired to estimate the cube root of 5 (≈ 1.7100).
   (a) Use Newton's method with a starting guess of $x_1 = 1$ and compute the next iterates x_2 and x_3.
   (b) Use the secant method with starting guesses $x_0 = 1$ and $x_1 = 2$ and compute the next iterates x_2 and x_3.
   (c) Which method do you expect will converge faster to the solution and why?

a) Newton

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x^2 - 5 = 0$

$f(x) = x^2 - 5$
$f'(x) = 2x$

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$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1-5}{2(1)} = 1 - \frac{-4}{2} = 1+2 = 3$$

$\boxed{x_2 = 3}$ ✗ $\quad x_1 = 1$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{9-5}{2(3)} = 3 - \frac{4}{6} = 3 - \frac{2}{3} = \frac{9}{3} - \frac{2}{3} = \frac{7}{3}$$

$f(3) = 9 - 5 = 4$
$f'(3) = 2(3) = 6$

$\boxed{x_3 = \frac{7}{3}}$ ✗

b) Secant

$$x_{n+1} = x_n + \frac{x_n - x_{n-1}}{\frac{f(x_{n-1})}{f(x_n)} - 1}$$

$f(x) = x^2 - 5$

$x_0 = 1$

$x_1 = 2$

$$f\left(\tfrac{7}{3}\right) = \frac{49}{9} - 5 = \frac{49}{9} - \frac{45}{9} = \frac{4}{9}$$

$$x_2 = x_1 + \frac{x_1 - x_0}{\frac{f(x_0)}{f(x_1)} - 1} = 2 + \frac{2-1}{\frac{f(1)}{f(2)} - 1} = 2 + \frac{1}{\frac{-4}{-1} - 1}$$

$$= 2 + \frac{1}{4-1} = 2 + \frac{1}{3} = \frac{6}{3} + \frac{1}{3} = \frac{7}{3}$$

$$\boxed{x_2 = 7/3}$$

$x_1 = 2$

$$x_3 = x_2 + \frac{x_2 - x_1}{\frac{f(x_1)}{f(x_2)} - 1} = \frac{7}{3} + \frac{7/3 - 2}{\frac{f(2)}{f(7/3)} - 1} = \frac{7}{3} + \frac{1/3}{\frac{-1}{4/9} - 1}$$

$$= \frac{7}{3} + \frac{1/3}{-9/4 - 1} = \frac{7}{3} + \frac{1/3}{-13/4} = \frac{7}{3} - \frac{4}{39} = \frac{91}{39} - \frac{4}{39}$$

$$\boxed{x_3 = \frac{87}{39}}$$

c) Newton will converge faster.

Newton method has quadratic convergence $e_{n+1} \rightarrow O(e_n^2)$ whereas

Secant method has superlinear convergence where

$e_{n+1} \Rightarrow O(e_n^{1.6})$

meaning that each iteration results in 1.6 times more correct decimal places,