Prof. Alan J. Laub May 2, 2011

EE 103 – MIDTERM EXAMINATION
Spring 2011

Instructions:

(a) The exam is closed-book (except for one page of notes) and will last 90 minutes.

(b) You may use a calculator if you are no longer able to perform arithmetic by hand but a calculator is not *needed* to solve any problem. Use of cell phones or any other electronic device is prohibited.

(c) Notation will conform as closely as possible to the standard notation used in the lectures.

(d) Do all 5 problems. Each problem is worth 20 points. Some partial credit may be assigned if warranted. Label clearly the problem number and the material you wish to be graded.

(e) Please put your full name, student number, and discussion section on the paper you turn in. Failure to include any of these will result in loss of 10 points.

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1. When doing partial pivoting, it turns out that, for the purposes of analyzing the method, we can assume that all row pivoting was done clairvoyantly at the beginning. This is illustrated below for the case of a 3×3 matrix. Here L_i^{-1} denotes an elementary ith step Gaussian elimination matrix and P_i denotes an ith step permutation matrix.

$$\begin{aligned} L_2^{-1} P_2 L_1^{-1} P_1 A &= L_2^{-1} P_2 L_1^{-1} (P_2^{-1} P_2) P_1 A \\ &= L_2^{-1} (P_2 L_1^{-1} P_2^{-1}) P_2 P_1 A \end{aligned}$$

where the matrix $P_2 L_1^{-1} P_2^{-1}$ is of the form of another first-step Gaussian elimination matrix, i.e., $\begin{bmatrix} 1 & 0 & 0 \\ \times & 1 & 0 \\ \times & 0 & 1 \end{bmatrix}$. Determine the values of the first column of this matrix.

2. If $A \in \mathbb{R}^{n\times n}$ is an arbitrary matrix, then computing A^2 in the usual way takes n^3 multiplications. However, if A has special structure, then this number can often be reduced substantially. Determine how many multiplications it takes to compute A^2 efficiently when A is

(a) upper bidiagonal (i.e., $a_{ij} = 0$ for $j - i > 1$ and $i > j$)

(b) a matrix of the form xy^T with x and y given nonzero n-vectors

(c) a matrix of the form xx^T with x a given nonzero n-vector

3. Let $A = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}$ where α is a nonnegative (possibly large) scalar.

(a) Compute $\kappa_1(A)$.

(b) It is a fact that if $A \in \mathbb{R}^{n\times n}_n$, then $A + E$ is nonsingular if $\|A^{-1}\| \, \|E\| < 1$ or, equivalently, if $\frac{\|E\|}{\|A\|} < \frac{1}{\kappa(A)}$. Stated differently, if $A + E$ is singular, then $\frac{\|E\|}{\|A\|} \geq \frac{1}{\kappa(A)}$, thereby giving a lower bound on $\kappa(A)$. In fact, it can be shown that

$$\frac{1}{\kappa(A)} = \min \left\{ \frac{\|E\|}{\|A\|} : A + E \;\text{ is singular} \right\}.$$

Therefore, $\kappa(A)$ is large if and only if there exists a singular matrix $A+E$ nearby in the sense that $\frac{\|E\|}{\|A\|}$ is small. For the specific matrix A in this problem, verify the formula above using the 1-norm by considering $E = \begin{bmatrix} -\frac{1}{1+\alpha} & \frac{1}{1+\alpha} \\ 0 & 0 \end{bmatrix}$.

4. (a) Write the *Lagrangian form* of the polynomial that interpolates the points $(1, 5), (2, 10), (3, 25)$. (Note there is no need to simplify the polynomial.)
   (b) Write the *power form* of the polynomial above, i.e., simplify the polynomial.

5. Suppose it is desired to estimate the cube root of 2 (≈ 1.2599).
   (a) Use Newton's method with a starting guess of $x_1 = 1$ and compute the next iterates x_2 and x_3.
   (b) Use the secant method with starting guesses $x_0 = 1$ and $x_1 = 2$ and compute the next iterates x_2 and x_3.
   (c) Which method do you expect will converge fastest to the solution and why?