Prof. Alan J. Laub March 17, 2008

EE 103 – FINAL EXAMINATION
Winter 2008

Instructions:

(a) The exam is closed-book (except for one page of notes) and will last 2 hours.

(b) You may use a calculator if you are no longer able to perform arithmetic by hand but you may not (and need not) use a calculator to solve any problem.

(c) Notation will conform as closely as possible to the standard notation used in the lectures.

(d) Do all 6 problems. Problems 1-5 are worth 15 points each, problem 6 is worth 25 points (5 points each). Some partial credit may be assigned if warranted.

(e) Label clearly the problem number and the material you wish to be graded.

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1. In a certain experiment, the following measurements (t_i, y_i) are taken:

$$(-2,4),\ (-1,0),\ (0,-2),\ (1,-1),\ (2,6)$$

It is desired to "fit" this data set with a function of the form

$$y = \beta_1 t^2 + \beta_2$$

where β_1 and β_2 are parameters to be determined. Compute the best parameters β_1 and β_2 using the method of least squares (by solving the normal equations since your calculations will be done by hand!).

2. Consider a Householder reflector of the form $H = I - \frac{2}{u^T u} u u^T$ for some nonzero vector $u \in \mathbb{R}^n$.

   (a) Show that $Hu = -u$.

   (b) It is known that a Householder reflector has $n-1$ eigenvalues at 1. What is the value of $\det(H)$? (Note: You must justify your answer carefully. Merely writing down a number will be worth 0 points.)

3. Compute $\kappa(2, A)(= \kappa(\lambda, A))$ for the matrix $A = \begin{bmatrix} 2 & 1000 \\ 0 & 3 \end{bmatrix}$.

4. Compute A^{100} for $A = \begin{bmatrix} 7 & 6 \\ -4 & -3 \end{bmatrix}$.

5. (a) A matrix $A \in \mathbb{R}^{m\times n}$ has SVD $A = U\Sigma V^T$ where U and V are orthogonal. What is an SVD of A^T?

   (b) A real $n \times n$ matrix $A = A^T > 0$ has an eigenvalue decomposition $A = X\Lambda X^T$ with $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ with all $\lambda_i > 0$ and X is orthogonal. What is an SVD of A?

   (c) A real $n \times n$ matrix $A = A^T < 0$ has an eigenvalue decomposition $A = X\Lambda X^T$ with $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ with all $\lambda_i < 0$ and X is orthogonal. What is an SVD of A?

6. In each case, choose the best answer, (A), (B), (C), (D), or (E).

   (a) The *characteristic equation* of a matrix $A \in \mathbb{R}^{n\times n}$ can also be defined by $\det(\lambda I - A) = 0$. Using this definition, which of the following is true?
   (A) it's always a polynomial of degree n (B) it's never a polynomial of degree n (C) this can't be the definition (D) this polynomial may have complex coefficients (E) none of these

   (b) If $A = U\Sigma V^T$ is an SVD of A then $A^+ =$
   (A) A^{-1} (B) $V\Sigma^+U^T$ (C) $U\Sigma^+V^T$ (D) $V\Sigma U^T$ (E) none of these

   (c) Which of the following matrices is not diagonalizable?
   (A) $\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (E) all are

   (d) The extrapolated Simpson's rule (the one implemented in `quadtx`) is employed to evaluate $\int_0^1 x^4 dx$. What is the value thus obtained?
   (A) $\frac{17}{90}$ (B) $\frac{1}{5}$ (C) $\frac{19}{90}$ (D) 0.2001 (E) none of these

   (e) The initial value problem $\dot{y} = 1 + t^2 + y$; $y(0) = 1$ is integrated via one step of Euler's method with step size $h = 0.01$. What is the result?
   (A) 1.01 (B) 1.00 (C) 0.98 (D) 1.02 (E) none of these