EE103
Applied Numerical Computing
Midterm Exam
October 25, 2007
(Closed-book examination, No calculators)

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Lname **Fname**

Question number	Score
1	25
2	20
3	25
4	21
Total	91

Problem 1:

The following data comprise the output from applying Newton's method for finding an $\bar{x}$ such that $f(\bar{x}) = 0$.

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1. To what value is the sequence of iterations converging?
2. What is your *estimate* of the rate of convergence, α? State why.
3. What is your *estimate* of the asymptotic error coefficient, λ? State why.
4. What can you say about the function's derivative, $f^{(1)}(\bar{x})$, at your limiting value of the x sequence? Why? What about $g_N^{(1)}(\bar{x})$?

```
1.626923076923077e+001
9.081360946745560e+000
5.437414656349567e+000
3.465672157837611e+000
2.375162859924642e+000
1.767842843999006e+000
1.428888726391136e+000
1.239587641978233e+000
1.133843976215715e+000
1.074771786275115e+000
1.041771307641842e+000
1.023335591519508e+000
1.013036460375617e+000
1.007282837243768e+000
1.004068567565098e+000
1.002272911177978e+000
1.001269765133417e+000
1.000709356138076e+000
1.000396282838074e+000
1.000221383983777e+000
1.000123676484484e+000
1.000069092047913e+000
1.000038598373044e+000
1.000021563037233e+000
1.000012046222108e+000
1.000006729639499e+000
1.000003759522892e+000
1.000002100262932e+000
1.000001173314941e+000
1.000000655474098e+000
        ⋮
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$$\frac{|x_{k+1} - 1|}{|x_k - 1|} = 2$$

① value goes to $\bar{x}=1$ ✓

②
Convergence is linear because as k gets large, $\frac{e_{k+1}}{e_k} \approx \frac{1}{2} < 1$

$\therefore \alpha = 1$ ✓

③ λ is estimated to around $\frac{1}{2}$ because when k is large,

$$\frac{x_{k+1}-1}{x_k-1} \approx \frac{1}{2} = \lambda$$ ✓

④ We know that for a newton's method sequence to converge NON-quadratically (linear in this case)

$(g_N'(1) \neq 0)$ $g_N'(\bar{x}) \neq 0$ that means

$$g_N'(\bar{x}) = \frac{f(\bar{x})f''(\bar{x})}{[f'(\bar{x})]} \neq 0$$ but numerator ✓

is zero so denominator has to be zero to satisfy inequality

$\therefore f'(\bar{x}) = 0$, $f'(1) = 0$

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Problem 2:

Assume $f(x)$ has a fourth continuous derivative. Show that

$$\frac{f(\bar{x}+h)-2f(\bar{x})+f(\bar{x}-h)}{h^2}$$

$$f(x) \cong f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \tfrac{1}{2}f''(\bar{x})(x-\bar{x})^2 + \tfrac{1}{6}f^{3}(\bar{x})(x-\bar{x})^3 + \tfrac{1}{24}f^{4}(\delta_{x,\bar{x}})(x-\bar{x})^4$$

is an $O(h^2)$ approximation for $f^{(2)}(\bar{x})$, the second derivative of f evaluated at $\bar{x}$.

[hint: Use Taylor's Theorem twice.]

Taylor's theorem

$$f(\bar{x}+h) = f(\bar{x}) + \cancel{f'(\bar{x})h} + \tfrac{1}{2}f''(\bar{x})h^2 + \cancel{\tfrac{1}{6}f'''(\bar{x})h^3} + \tfrac{1}{24}f^{4}(\delta)h^4$$

$$+\quad f(\bar{x}-h) = f(\bar{x}) - \cancel{f'(\bar{x})h} + \tfrac{1}{2}f''(\bar{x})h^2 - \cancel{\tfrac{1}{6}f'''(\bar{x})h^3} + \tfrac{1}{24}f^{4}(\delta)x^4$$

$$f(\bar{x}+h) + f(\bar{x}-h) - 2f(\bar{x}) = 0 + f''(\bar{x})h^2$$

$$\boxed{\frac{f(\bar{x}+h)+f(\bar{x}-h)-2f(\bar{x})}{h^2} = f''(\bar{x}) + \tfrac{1}{24}f^{4}(\delta)h^2}$$

$O(h^2)$ term

$$\lim_{h\to 0}\left|\frac{\dfrac{f(\bar{x}+h)+f(\bar{x}-h)-2f(\bar{x})}{h^2}}{h^2}\right| = \left|\tfrac{1}{24}f(\delta)\right|$$

$$\therefore O(h^2)$$

Problem 3:
Suppose we're using Newton's method for finding a solution of the equation $f(x)=0$. Suppose the method works and we have $x_k \to \bar{x}$ and we also know that $\bar{x}$ is a root of multiplicity 4. We're interested in studying the rate of convergence. To make matters specific, let's consider $f(x)=x^4$ (of course, we don't need an algorithm for this problem; our point is to understand rates), where we begin Newton's method with the positive value $x_0 = 1$.

(a) What is the rate of convergence of Newton's method, $x \leftarrow x - f(x)/f'(x)$, in this case? What is the value of the asymptotic error coefficient, λ ?

(b) In place of Newton's method, suppose we use $x \leftarrow x - 2f(x)/f'(x)$. What is the rate of convergence of this method? What is the value of the asymptotic error coefficient, λ ? How many iterations will it take to be within 10^{-3} of the root (An expression for the answer is sufficient).

(c) In place of the method of (b), suppose we use $x \leftarrow x - 4f(x)/f'(x)$. How many iterations will it take to converge?

(d) Now suppose that $f(x) = P_n(x)$, an n^{th} degree polynomial with n distinct roots. As above, Newton's method produces a sequence $x_k \to \bar{x}$. Show whether or not the convergence rate is at least quadratic.

Hint: Consider the facts

$$P_n(x) = (x-\bar{x})P_{n-1}(x),\ P_n^{(1)}(x) = ?$$

$$g_N^{(1)}(x) = \frac{P_n(x)P_n^{(2)}(x)}{(P_n^{(1)}(x))^2}$$

a) $f(x) = x^4 \quad f'(x) = 4x^3$ 9/5

$g_N(x) = x - \frac{x^4}{4x^3} = x - \frac{1}{4}x = \frac{3}{4}x$ ✓

$g_N'(x) = \boxed{\frac{3}{4}} \leq 1 \Rightarrow \boxed{\alpha = 1}$

$\boxed{\text{R.O.C. is linear. with } \lambda = \frac{3}{4}}$

b) $g_{N2}(x) = x - 2\frac{f(x)}{f'(x)}$

$= x - \cancel{2}\left(\frac{x^4}{\cancel{4}_2 x^3}\right) = x - \frac{1}{2}x = \frac{1}{2}x$

$g_{N2}'(x) = \frac{1}{2} < 1$ $\alpha = 1$ 5/5

$\boxed{\text{R.o.c. is linear and } \lambda = \frac{1}{2}}$

$\lim_{k\to\infty} \left|\frac{e_{k+1}}{e_k}\right| = \frac{1}{2}$ large k

$e_{k+1} \approx \frac{1}{2} e_k$

$\frac{e_{k+1}}{e_k} \approx \frac{1}{2}$ 5/5

$\left(\frac{1}{2}\right)^k = 10^{-3}$ to find iterations to reach 10^{-3} within root

5/5

c) $g_{N3} = x - 4\left(\frac{f(x)}{f'(x)}\right) \Rightarrow g_{N3}(x) = 0 \Rightarrow x = 0$ no matter what value

$x = 0$ super linear conv., it would take only one iteration to converge since $x = 0$ no matter what

d) $f(x) = P_n(x)$, n distinct roots

$P_n(x) = (x - \bar{x})P_{n-1}(x)$

$P_n'(x) = P_{n-1}(x) + (x - \bar{x})P_{n-1}'(x) \Rightarrow P_n'(\bar{x}) = P_{n-1}(x)$

$g_N'(\bar{x}) = \frac{\overset{0}{\cancel{P_n(\bar{x})}} P_n''(\bar{x})}{(P_n'(\bar{x})^2)}$

From above non-zero

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Since $P_n(\bar{x})=0$ and $P_n'(\bar{x})\neq 0$

$\Rightarrow g_N'(x)=0$

and since $g_n'(\bar{x})$ is continuous and

cond it exists (because it's a polynomial)

at lest

$\therefore \alpha$ is atleast 2 $\Rightarrow$ quadratic

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Problem 4: $1+100xe^{-50x^2}$

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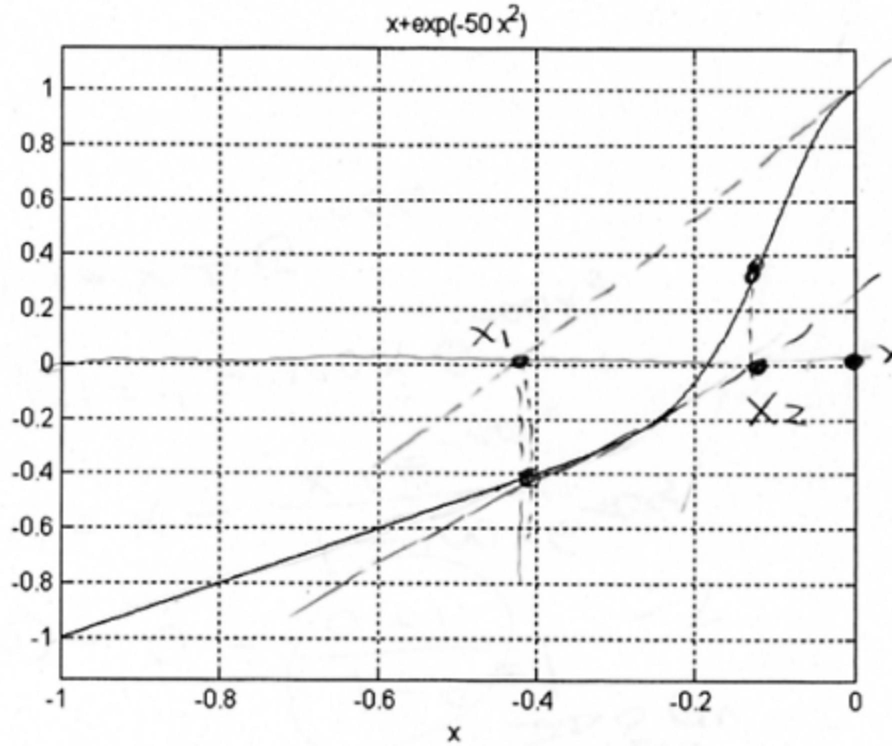
(a) Let $f(x)=x+e^{-50x^2}$. There is a unique solution, $\bar{x}$, for $f(x)=0$ in the interval $[-1,0]$.

(i) The plot of $f(x)$ is given below. By inspecting the graph, what can you say about the rate of convergence if Newton's method converges to the solution? Why?

(ii) Starting with $x_0=0$, mark the approximate locations, with x_1 and x_2, where those values would be if we had implemented Newton's method.



(iii) Now, implement Newton's method numerically, starting with $x_0=0$ as the initial point. Also, assume we are implementing Newton's method on a machine that uses the IEEE double-precision standard.

Compute the first 5 numbers generated from the Newton iterations, starting with $x_0=0$. (**Hint**: use $e^{-50}\approx 1.9287\times10^{-22}$, and recall that $\varepsilon_M\approx 1.1102\times10^{-16}$ for an IEEE double-precision machine).

(iv) What do you observe from part (iii)? Explain.

i) convergence is quadratic since

$f'(\bar{x}) \neq 0 \Rightarrow g_N'(\bar{x}) = 0$, obviously

$\therefore$ we know $\alpha \geq 2 \rightarrow$ atleast quadratic

$\therefore \alpha = 2 \Rightarrow$ quadratic ✓

(ii) $f(x) = x + e^{-50x^2}$

$f'(x) = 1 - 100x\,e^{-50x^2}$

$$g(x) = x - \frac{x + e^{-50x^2}}{1 - 100x e^{-50x^2}}$$

$x_1 = g(0) = 0 - \frac{0+1}{1-0} = 1$ -2

$x_2 = g(1) = 1 - \frac{1 + \cancel{e^{-50}}}{1 - \cancel{100 e^{-50}}}$ → 0 for IEEE precision mach. → 1

$g(1) = 0$ and so on...

$x_0 = 0$
$x_1 = 1$
$x_2 = 0$ -2
$x_3 = 1$
$x_4 = 0$
$x_5 = 1$

(iv) x never converges to a value in part iii because a ~~IEEE~~ precision machine sees e^{-50} and $100e^{-50}$ as zero since both numbers are smaller than ε_m so 1 and 0 will alternate.