EE103
Applied Numerical Computing
Midterm Exam
October 25, 2007
(Closed-book examination, No calculators)

Name: [redacted]______ **Student ID Number:** [redacted]
Lname **Fname**

Question number	Score
1	19
2	20
3	22.5
4	3
Total	64.5

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Problem 1:

The following data comprise the output from applying Newton's method for finding an $\bar{x}$ such that $f(\bar{x}) = 0$.

1. To what value is the sequence of iterations converging?

2. What is your *estimate* of the rate of convergence, α? State why.

3. What is your *estimate* of the asymptotic error coefficient, λ? State why.

4. What can you say about the function's derivative, $f^{(1)}(\bar{x})$, at your limiting value of the x sequence? Why? What about $g_N^{(1)}(\bar{x})$?

1st term 1.626923076923077e+001
2nd term 9.081360946745560e+000
3rd term 5.437414656349567e+000
3.465672157837611e+000
2.375162859924642e+000
1.767842843999006e+000
1.428888726391136e+000
1.239587641978233e+000
1.133843976215715e+000
1.074771786275115e+000
1.041771307641842e+000
1.023335591519508e+000
1.013036460375617e+000
1.007282837243768e+000
1.004068567565098e+000
1.002272911177978e+000
1.001269765133417e+000
1.000709356138076e+000
1.000396282838074e+000
1.000221383983777e+000
1.000123676484484e+000
1.000069092047913e+000
1.000038598373044e+000
1.000021563037233e+000
1.000012046222108e+000
1.000006729639499e+000
1.000003759522892e+000
1.000002100262932e+000
1.000001173314941e+000
1.000000655474098e+000
⋮

① It appears that the iterations' values are approaching 1. That's the trend in this problem.

② say $\bar{x} = 1$ $\quad \frac{x_{k+1} - \bar{x}}{|x_k - \bar{x}|^\alpha} = \lambda$

$\frac{|1.62 \cdot 10^1 - 1|}{|9.08 - 1|^\alpha} = \frac{15.27}{8.08^\alpha} = \lambda = \frac{|9.08 - 1|}{|5.44 - 1|^\alpha} = \frac{8.08}{4.44^\alpha}$

$15.27 \cdot 4.44^\alpha = 8.08 \cdot 8.08^\alpha$

If $\alpha = 1$ $\quad \approx 62 \quad = 64$

If $\alpha = 2$ $\quad 16 \cdot 16 \quad = 8 \cdot 64$
$\quad = 40 \quad = 522$

$\alpha = 1$ produces the best estimate and satisfies both sides of eq. Since the equation for λ produces a consistent λ for the 2nd & third term when $\alpha = 1$.

③ based off of $\alpha = 1$, we can use the equation: $\frac{x_{k+1} - \bar{x}}{|x_k - \bar{x}|^\alpha} = \lambda$

let $\bar{x} = 1$, $\alpha = 1$

$\frac{|16.27 - 1|}{|9.08 - 1|^1} = \lambda = \frac{15.27}{8.08} \approx 1.9$

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$8.08 \overline{)15.27}$ = 1.9

(4) Since it converges linearly,

$f(\bar{x}) = 0$

$g_N'(\bar{x}) \neq 0$

Problem 2:

Assume $f(x)$ has a fourth continuous derivative. Show that

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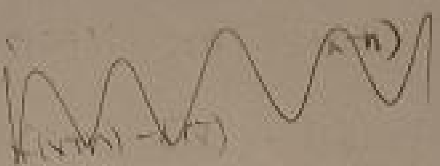
f $$\frac{f(\bar{x}+h)-2f(\bar{x})+f(\bar{x}-h)}{h^2}$$

is an $O(h^2)$ approximation for $f^{(2)}(\bar{x})$, the second derivative of f evaluated at $\bar{x}$.

[hint: Use Taylor's Theorem twice.]



$$-\left(f(\bar{x})-f(\bar{x}+h)+f(\bar{x})-f(\bar{x}-h)\right)$$

$$f(x)=f(\bar{x})+f'(\bar{x})(x-\bar{x})+\tfrac{1}{2}f''(\bar{x})(x-\bar{x})^2+\tfrac{1}{6}f'''(\xi)(x-\bar{x})^3$$

$$f(\bar{x}+h)=f(\bar{x})+f'(\bar{x})(x-\bar{x})+\tfrac{1}{2}f''(\bar{x})(x-\bar{x})^2+\tfrac{1}{6}f'''(\xi_1)(x-\bar{x})^3$$

$$f(\bar{x}-h)=f(\bar{x})-f'(\bar{x})(x-\bar{x})+\tfrac{1}{2}f''(\bar{x})(x-\bar{x})^2-\tfrac{1}{6}f'''(\xi_2)(x-\bar{x})^3$$

$$f(\bar{x}+h)+f(\bar{x}-h)=2f(\bar{x})+f''(\bar{x})(x-\bar{x})^2+\tfrac{1}{6}f'''(\xi_1)(x-\bar{x})^3-\tfrac{1}{6}f'''(\xi_2)(x-\bar{x})^3$$

$$\frac{f(\bar{x}+h)+f(\bar{x}-h)-2f(\bar{x})}{(x-\bar{x})^2}=f''(\bar{x})+\tfrac{1}{6}f'''(\xi_1)(x-\bar{x})+\tfrac{1}{6}f'''(\xi_2)(x-\bar{x})$$

these two don't cancel!

let $(x-\bar{x})=h$ & $h\to 0$

$$\Rightarrow \frac{f(\bar{x}+h)+f(\bar{x}-h)-2f(\bar{x})}{h^2}=f''(\bar{x})+0=f''(\bar{x})$$

∴ since the expression converges to a constant value which is $f''(\bar{x})$ at $\bar{x}$, we can say that $\frac{f(\bar{x}+h)+f(\bar{x}-h)-2f(\bar{x})}{h^2}$ is $O(h^2)$ for $f''(\bar{x})$.

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Problem 3:
Suppose we're using Newton's method for finding a solution of the equation $f(x)=0$. Suppose the method works and we have $x_k \to \bar{x}$ and we also know that $\bar{x}$ is a root of multiplicity 4. We're interested in studying the rate of convergence. To make matters specific, let's consider $f(x)=x^4$ (of course, we don't need an algorithm for this problem; our point is to understand rates), where we begin Newton's method with the positive value $x_0 = 1$.

(a) What is the rate of convergence of Newton's method, $x \leftarrow x - f(x)/f'(x)$, in this case? What is the value of the asymptotic error coefficient, λ ?

(b) In place of Newton's method, suppose we use $x \leftarrow x - 2f(x)/f'(x)$. What is the rate of convergence of this method? What is the value of the asymptotic error coefficient, λ ? How many iterations will it take to be within 10^{-3} of the root (An expression for the answer is sufficient).

(c) In place of the method of (b), suppose we use $x \leftarrow x - 4f(x)/f'(x)$. How many iterations will it take to converge?

(d) Now suppose that $f(x) = P_n(x)$, an n^{th} degree polynomial with n distinct roots. As above, Newton's method produces a sequence $x_k \to \bar{x}$. Show whether or not the convergence rate is at least quadratic.

Hint: Consider the facts

$$P_n(x) = (x-\bar{x})P_{n-1}(x),\ P_n^{(1)}(x) = ?$$
$$g_N^{(1)}(x) = \frac{P_n(x)P_n^{(2)}(x)}{(P_n^{(1)}(x))^2}$$

(a) $f(x)=x^4$ $\quad x_0=1$ $\qquad f'(x)=4x^3 \Rightarrow f'(\bar{x})=f'(0)=0$

$\bar{x}=0$

$g_N(x)=x-\frac{f(x)}{f'(x)}=x-\frac{x^4}{4x^3}=x-\frac{x}{4}=\frac{3x}{4}$

$g_N'(x)=\frac{3}{4}$ $\quad g_N'(\bar{x})=\frac{3}{4}$

$\alpha=1$ $\quad \lambda=|g_N'(\bar{x})|=\frac{3}{4}\le 1$

$g_N''(\bar{x})=0$

since $g_N'(\bar{x})\neq 0$ $\quad f'(\bar{x})=0$

$g_N''(\bar{x})$ = exists & cont.

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we can say that $\alpha=1$, $\lambda=\frac{3}{4}$, linear convergence

↑ ROC

(b) $\bar{x}=0$

$g_N(x)=x-\frac{2f(x)}{f'(x)}=x-\frac{2x^4}{4x^3}=x-\frac{x}{2}=\frac{1}{2}x$ $\qquad f'(\bar{x})=f'(0)=0$

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$g_N'(x)=\frac{1}{2} \Rightarrow g_N'(\bar{x})=\frac{1}{2} \Rightarrow \alpha=1, \lambda=|g_N'(\bar{x})|=\frac{1}{2}$

$g_N''(\bar{x})=0$

since $g_N'(\bar{x})\neq 0$, $f'(\bar{x})=0$

$g_N'(\bar{x})$ = exists & cont.

we can say that ROC: $\alpha=1$, $\lambda=\frac{1}{2}$, & it converges linearly.

$\frac{|x_{k+1}-\bar{x}|}{|x_k-\bar{x}|^\alpha}=\frac{1}{2}=\frac{x_{k+1}-\bar{x}}{x_k-\bar{x}}$

$\Rightarrow \frac{1}{2^n}=|0-1\cdot 10^{-3}|$ where n = # of iterations

let $x_{k+1}=1$ $\qquad x_k-\bar{x}=2(x_{k+1}-\bar{x})$

$\frac{1}{2^n}=0.001$

$1000=2^n$

5/5 $\quad n=\log_2(1000)$ iterations.

(c) $g_N(x)=x-\frac{4f(x)}{f'(x)}=x-\frac{4x^4}{4x^3}=0 \Rightarrow g_N'(\bar{x})=0$ $\qquad$ 2.5/5

$\bar{x}=0$ $\quad g_N'(x)=0$

$g_N''(x)=0$

The sequence will not converge according to the method used since $g_N(\bar{x})=0$, $g_N'(\bar{x})=0$, $g_N''(\bar{x})=0$

$\Rightarrow g_N'(\bar{x})=0$

$g_N''(\bar{x})=0$

No # of iterations will make the expression converge.

$p_n'(\bar{x}) = p_{n-1}(\bar{x}) + \cancel{(x-\bar{x})p_{n-1}'(x)}$

$p_n'(\bar{x}) = p_{n-1}(\bar{x})$

$p_n''(x) = p_{n-1}'(x) + p_{n-1}(x) + (x-\bar{x})p_{n-1}(x)$

d

$p_n(x) = (x-\bar{x})p_{n-1}(x)$

$p_n'(x) = p_{n-1}(x) + (x-\bar{x})p_{n-1}(x)$

$p_n'(\bar{x}) = p_{n-1}(\bar{x}) + \cancel{(\bar{x}-\bar{x})}p_{n-1}(\bar{x}) = p_n(\bar{x}) \neq 0$

$$g_n'(x) = \frac{p_n(x)p_n''(x)}{p_n'(x)^2} = \frac{(x-\bar{x})p_{n-1}(x)\left[p_{n-1}'(x) + p_{n-1}(x) + (x-\bar{x})p_{n-1}(x)\right]}{p_{n-1}(x)^2 + 2p_{n-1}(x)^2(x-\bar{x}) + (x-\bar{x})^2 p_{n-1}(x)^2}$$

$\bar{x} = 0$

$$g_n'(\bar{x}) = \frac{(\bar{x}-\bar{x})p_{n-1}(\bar{x})\left[p_{n-1}'(\bar{x}) + p_{n-1}(\bar{x}) + (\bar{x}-\bar{x})p_{n-1}(\bar{x})\right]}{p_{n-1}(\bar{x})^2 + 2p_{n-1}(\bar{x})^2(\bar{x}-\bar{x}) + (\bar{x}-\bar{x})^2 p_{n-1}(\bar{x})^2}$$

$$= \frac{0}{p_{n-1}(\bar{x})^2} = 0$$

$$g_n''(x) = \frac{p_n''(x)}{p_n'(x)} = \frac{p_{n-1}'(x) + p_{n-1}(x) + (x-\bar{x})p_{n-1}(x)}{p_{n-1}(x) + (x-\bar{x})p_{n-1}(x)}$$

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$$g_n''(\bar{x}) = \frac{p_{n-1}'(\bar{x}) + p_{n-1}(\bar{x}) + (\bar{x}-\bar{x})p_{n-1}(\bar{x})}{p_{n-1}(\bar{x}) + (\bar{x}-\bar{x})p_{n-1}(\bar{x})}$$

$$g_n''(\bar{x}) = \frac{p_{n-1}'(\bar{x}) + p_{n-1}(\bar{x})}{p_{n-1}(\bar{x})} \neq 0$$

we can see that $g_n''(\bar{x})$ exists & is continuous.

$\therefore$ since $p_n'(\bar{x}) \neq 0$

$g_n'(\bar{x}) = 0$

$g_n''(\bar{x}) \neq 0$

$g_n''(\bar{x})$ = cont. & exists

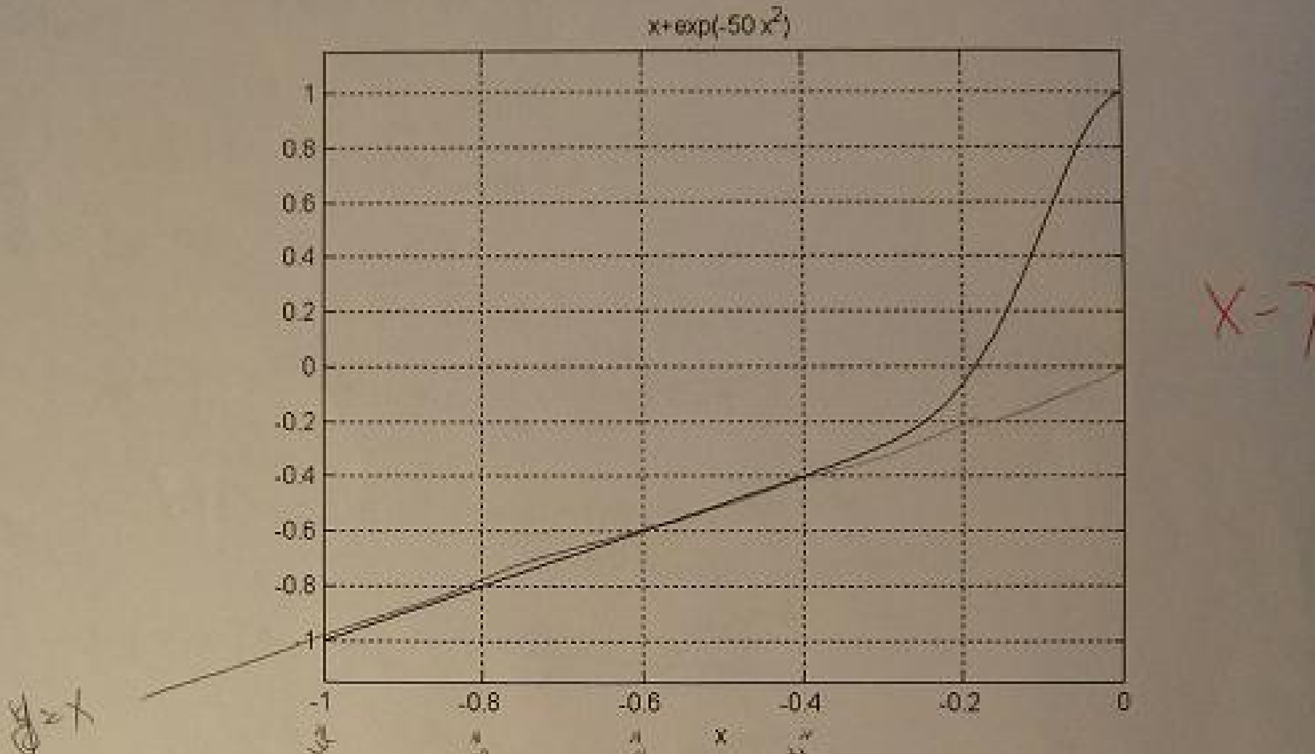
$\Rightarrow \alpha = 2$, at least quadratic convergence.

Problem 4:

(a) Let $f(x) = x + e^{-50x^2}$. There is a unique solution, $\bar{x}$, for $f(x) = 0$ in the interval $[-1, 0]$.

(i) The plot of $f(x)$ is given below. By inspecting the graph, what can you say about the rate of convergence if Newton's method converges to the solution? Why?

(ii) Starting with $x_0 = 0$, mark the approximate locations, with x_1 and x_2, where those values would be if we had implemented Newton's method.



(iii) Now, implement Newton's method numerically, starting with $x_0 = 0$ as the initial point. Also, assume we are implementing Newton's method on a machine that uses the IEEE double-precision standard.

Compute the first 5 numbers generated from the Newton iterations, starting with $x_0 = 0$.

(**Hint**: use $e^{-50} \approx 1.9287 \times 10^{-22}$, and recall that $\varepsilon_M \approx 1.1102 \times 10^{-16}$ for an IEEE double-precision machine).

(iv) What do you observe from part (iii)? Explain.

(i) The rate of convergence should be $\alpha = 1$ or linear convergence because from $[-1, -0.4]$ the slope of the line is 1, $g_N'(x) = 1$ which means $g_N'(x) \neq 0$, so the convergence has to be linear with ROC $\alpha = 1$.

in the interval $[-1, -0.4]$

X -6

(ii) marked on x axis of graph

(iii) $g_N(x) = x - \dfrac{x + e^{-50x^2}}{1 + -100x e^{-50x^2}} =$ ✓

$g_N(x_0) = g_N(0) = 0 - \dfrac{0 + 1}{1 + 0} = -1$ ✓

$g_N(x_1) = g_N(0.2) = -0.2 - \dfrac{-0.2 + (e^{-50})^{0.04}}{1 + 20(e^{-50})^{0.04}} = -0.2 - \dfrac{-0.2 + (1.9287 \cdot 10^{-22})^{0.04}}{1 + 20 \cdot (1.9287 \cdot 10^{-22})^{0.04}}$

$g_N(-0.4) = -0.4 - \dfrac{-0.4 + (1.9287 \cdot 10^{-22})^{0.16}}{1 + 40(1.9287 \cdot 10^{-22})^{0.16}}$

? -5

$g_N(-0.6) = -0.6 - \dfrac{-0.6 + (1.9287 \cdot 10^{-22})^{0.36}}{1 + 60(1.9287 \cdot 10^{-22})^{0.36}}$

$g_N(-0.8) = -0.8 - \dfrac{-0.8 + (1.9287 \cdot 10^{-22})^{0.64}}{1 + 80(1.9287 \cdot 10^{-22})^{0.64}}$

$g_N(-1) = -1 - \dfrac{-1 + (1.9287 \cdot 10^{-22})}{1 + 100(1.9287 \cdot 10^{-22})}$

(iii) The iterations seem to converge to the value 0 because ~~tee~~ if we take in account of precision of IEEE machine $e^{-50} = 0$ so $g_N(\bar{x}) = 0$.

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