EE 103
Applied Numerical Computing
Instructor: Lara Dolecek

Fall 2013 Midterm A
Wednesday, November 6, 2013

Maximum score is 100 points. You have 110 minutes to complete the exam. Please show your work. Good luck!

Your Name: Emily Im

Your ID Number: 404 050 824

Name of person on your left: Jeannie Nguyen

Name of person on your right: David AYVAZYAN

Problem	Score	Possible
1	14	15
2	15	15
3	30	40
4	10	10
5	10	10
6	10	10
Total	89	100

1. (5 + 10 pts)

5

(a) Consider two n-vectors, x and y. When is the inner product of x and y exactly 0 ?

(b) Express the norm $\|x+y\|$ in terms of $\|x\|$, $\|y\|$, and the angle θ between x and y.

(a) $x^T y = 0$ when $x \perp y$ (b/c of $x^T y = \|x\|\|y\| \cos\theta = 0$ when $\theta = \frac{n\pi}{2}$)

(b) $\|x+y\| = \sqrt{(x+y)^T(x+y)} = \sqrt{x^Tx + x^Ty + y^Tx + y^Ty}$

$\|x\|^2 = x^T x$

$= \sqrt{\|x\|^2 + 2x^Ty + \|y\|^2}$

$\cos\theta = \dfrac{x^T y}{\|x\| \|y\|}$

$x^T y = \|x\| \|y\| \cos\theta$

$\boxed{\sqrt{\|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \cos\theta}}$

9

2. (5+10 pts) Consider the following matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & a_1 \\ 0 & 0 & \cdots & 1 & 0 & a_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & a_{n-2} \\ 1 & 0 & \cdots & 0 & 0 & a_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & a_n \end{bmatrix} \tag{1}$$

(a) When is the matrix A non-singular ? $AB = I$

(b) Assuming conditions in part (a) hold, compute the inverse of A.

(a) A is nonsingular when $a_n \neq 0$, $a_1, \ldots, a_{n-1}$ can be of any values.

(b) $\begin{bmatrix} 0 & 1 & a_1 \\ 1 & 0 & a_2 \\ 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 1 & -a_2/a_3 \\ 1 & 0 & -a_1/a_3 \\ 0 & 0 & 1/a_3 \end{bmatrix}$

$a_3 b_{31} = 0 \Rightarrow b_{31} = 0 \qquad a_3 b_{32} = 0 \Rightarrow b_{32} = 0 \qquad a_3 b_{33} = 1 \Rightarrow b_3 = \frac{1}{a_3}$

$b_{21} + \cancel{a_1 b_{31}} = 1 \qquad b_{22} + \cancel{a_1 b_{32}} = \emptyset \qquad b_{23} + a_1 b_{33} = 0$

$b_{21} = 1 \qquad b_{22} = 0 \qquad b_{23} + \frac{a_1}{a_3} = 0$

$b_{23} = -\frac{a_1}{a_3}$

$b_{11} + \cancel{a_2 b_{33}} = 0 \qquad b_{12} + \cancel{a_2 b_{32}} = 1 \qquad b_{13} + a_2 b_{33} = 0$

$b_{11} = 0 \qquad b_{12} = 1 \qquad b_{13} + \frac{a_2}{a_3} = 0$

$b_{13} = -\frac{a_2}{a_3}$

$$\boxed{A^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & -a_{n-1}/a_n \\ 0 & 0 & \cdots & 1 & 0 & -a_{n-2}/a_n \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & -a_2/a_n \\ 1 & 0 & \cdots & 0 & 0 & -a_1/a_n \\ 0 & 0 & \cdots & 0 & 0 & 1/a_n \end{bmatrix}}$$

$[1 \times 3][3\times 3] = [1\times 3]$

3. (10 + 15 +10+ 5 pts) Consider matrix A,

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 8 & 2 \\ 2 & 2 & 6 \end{bmatrix} \quad (2)$$

$\frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

(a) Is A positive definite ? Prove or disprove.

(b) Factorize A as $A = LL^T$ where L is lower triangular. Please show all steps.

(c) Solve $Ax = b$ for $b = [4\ 0\ 2]^T$ using forward and backward substitution. Solve $Lw=b$ Then $L^Tx = w$

(d) How many total flops does it take to solve for $Ax = b$ in the previous part (account for the cost of factorization).

(a) $[x_1\ x_2\ x_3]\begin{bmatrix} 2 & 2 & 2 \\ 2 & 8 & 2 \\ 2 & 2 & 6 \end{bmatrix} = [2x_1+2x_2+2x_3 \quad 2x_1+8x_2+2x_3 \quad 2x_1+2x_2+6x_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$2x_1^2 + 2x_1 2x_2 + 2x_1x_3 + 2x_1x_2 + 8x^2 + 2x_2x_3 + 2x_1x_3 + 2x_2x_3 + 6x_3^2 > 0$

$2x_1^2 + 8x_2^2 + 6x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 > 0$

$(x_1+x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2 \quad (x_2+x_3)^2 = x_2^2 + 2x_2x_3^2 + x_3^2 \quad (x_1+x_3)^2 = x_1^2 + 2x_1x_3 + x_3^2$

$(x_1+x_2)^2 + (x_2+x_3)^2 + (x_1+x_3)^2 + 6x_2^2 + 4x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 > 0$

$(x_1+x_2)^2 + 2(x_2+x_3^2) + (x_1+x_3)^2 + 5x_2^2 + 3x_3^2 > -2x_1x_2 = 2x_1x_3 \Longrightarrow \boxed{A \text{ is P.D.}}$

A is symmetric, nonsingular, with (+) diagonal entries

−5

(b) $l_{11} = \sqrt{a_{11}} \quad L_{21} = \frac{1}{l_{11}}A_{21} \quad L_{22}L_{22}^T = A_{22} - L_{21}L_{21}^T$

① $\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{2} & & \\ \sqrt{2} & & \end{bmatrix}$ 1 flop, 2 flops

$\begin{bmatrix} 8 & 2 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}[\sqrt{2}\ \sqrt{2}] = \begin{bmatrix} 8 & 2 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$ 4 flops 4 flops

② $\begin{bmatrix} \sqrt{6} & 0 \\ 0 & \end{bmatrix}$ 1 flop $\quad 4-(0)(0)=4 \quad \sqrt{4}=2$ 1 flop

$$\boxed{A = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{2} & \sqrt{6} & 0 \\ \sqrt{2} & 0 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

(c) $Lw=b \quad \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{2} & \sqrt{6} & 0 \\ \sqrt{2} & 0 & 2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

$\sqrt{2}w_1 = 4 \Rightarrow w_1 = \frac{4}{\sqrt{2}}$ (1 flop)

$\sqrt{2}\cdot\frac{4}{\sqrt{2}} + w_2\sqrt{6} = 0 \Rightarrow w_2\sqrt{6} = -4 \quad w_2 = \frac{-4}{\sqrt{6}}$ (3 flops)

$\sqrt{2}\cdot\frac{4}{\sqrt{2}} + w_3 2 = 2 \Rightarrow 2w_3 = -2 \quad w_3 = -1$ (3 flops)

$L^Tx = w$

$\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{2} \\ -4/\sqrt{6} \\ -1 \end{bmatrix}$ $\to 2\sqrt{2}$

$2x_3 = -1 \quad x_3 = -\frac{1}{2}$ (1 flop)

$\sqrt{6}x_2 = \frac{-4}{\sqrt{6}} \quad x_2 = -\frac{4}{6}$ (1 flop)

$\sqrt{2}x_1 - \frac{4\sqrt{2}}{6} - \frac{\sqrt{2}}{2} = 2\sqrt{2}$ (5 flops)

$\sqrt{2}x_1 - \frac{16\sqrt{2}}{24} - \frac{12\sqrt{2}}{24} = \frac{48\sqrt{2}}{24}$

$\sqrt{2}x_1 = \frac{48-28\sqrt{2}}{24} = x_1 = \frac{20}{24}$

$$\boxed{x = \begin{bmatrix} 5/6 \\ -2/3 \\ -1/2 \end{bmatrix}}$$

−2

(d) Cholesky 13 flops Forward 7 flops Backward 7 flops $\Rightarrow \boxed{27 \text{ flops}}$

−3

4. (10 pts) True or False.

10

Circling the correct answer is worth +2 points, circling an incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) Every triangular matrix is non-singular.

TRUE FALSE

(b) Schur complement of a positive definite matrix is positive definite.

TRUE FALSE

(c) All positive semi-definite matrices have strictly positive diagonal entries.

TRUE FALSE

(d) The norm $\|x\|$ of a unit vector x is zero.

TRUE FALSE

(e) Every matrix has a unique left inverse.

TRUE FALSE

5. (10 pts) For what values of a is the following matrix A positive definite ? For what values of a is the following matrix A positive semi-definite ? Please show all steps.

$$A = \begin{bmatrix} a & 0 & \cdots & 0 & 0 \\ 0 & a & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a & 0 \\ 0 & 0 & \cdots & 0 & a \end{bmatrix}$$

$$x^T A = [x_1 \ x_2 \cdots x_{n-1} \ x_n] \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & a & 0 \\ & & & 0 & a \end{bmatrix} = [ax_1 \ ax_2 \cdots ax_{n-1} \ ax_n]$$

$$x^T A x = [ax_1 \ ax_2 \cdots ax_{n-1} \ ax_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = ax_1^2 + ax_2^2 + \cdots + ax_{n-1}^2 + ax_n^2$$

$$x^T A x = a \underbrace{(x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^2)}_{\text{always positive b/c each term is squared}}$$

Positive definite

$x^T A x > 0$ $\quad \boxed{P.D. \Rightarrow a > 0}$

Positive semidefinite

$x^T A x \geq 0$ $\quad \boxed{P.S.D. \Rightarrow a \geq 0}$

6. (10 pts) Find the least norm solution for the following system of equations,

$\hat{x} = A^T(AA^T)^{-1}b$

$$\begin{cases} x_3 = 5 \\ x_1 + 2x_2 + 2x_4 = 0 \\ 4x_1 - x_2 - x_4 = 10. \end{cases}$$

10

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 2 \\ 4 & -1 & 0 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ 0 \\ 10 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 2 \\ 4 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+4+4 & 4-2-2 \\ 0 & 4-2-2 & 16+1+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 9 & 0 & 0 & 1 & 0 \\ 0 & 0 & 18 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{18} \end{array}\right] \quad (AA^T)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{18} \end{bmatrix}$$

$$(AA^T)^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{18} \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ \frac{10}{18} \end{bmatrix}$$

$$A^T(AA^T)^{-1}b = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ \frac{5}{9} \end{bmatrix} = \begin{bmatrix} 20/9 \\ -5/9 \\ 5 \\ -5/9 \end{bmatrix}$$

$$\boxed{\hat{x} = \begin{bmatrix} 20/9 \\ -5/9 \\ 5 \\ -5/9 \end{bmatrix}}$$

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Total	89	100

1. (5 + 10 pts)

(a) Consider two n-vectors, x and y. When is the inner product of x and y exactly 0 ?

(b) Express the norm $\|x+y\|$ in terms of $\|x\|$, $\|y\|$, and the angle θ between x and y.

5

(a) $x^T y = 0$ when $x \perp y$ (b/c of $x^T y = \|x\|\|y\| \cos\theta = 0$ when $\theta = \frac{n\pi}{2}$)

(b) $\|x+y\| = \sqrt{(x+y)^T(x+y)} = \sqrt{x^T x + x^T y + y^T x + y^T y}$

$\|x\|^2 = x^T x$

$= \sqrt{\|x\|^2 + 2x^T y + \|y\|^2}$

$\cos\theta = \frac{x^T y}{\|x\| \|y\|}$

$\boxed{\sqrt{\|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \cos\theta}}$

$x^T y = \|x\|\|y\| \cos\theta$

9

2. (5+10 pts) Consider the following matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & a_1 \\ 0 & 0 & \cdots & 1 & 0 & a_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & a_{n-2} \\ 1 & 0 & \cdots & 0 & 0 & a_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & a_n \end{bmatrix} \tag{1}$$

(a) When is the matrix A non-singular ? $AB = I$

(b) Assuming conditions in part (a) hold, compute the inverse of A.

(a) A is nonsingular when $a_n \neq 0$, $a_1, \ldots, a_{n-1}$ can be of any values.

(b) $\begin{bmatrix} 0 & 1 & a_1 \\ 1 & 0 & a_2 \\ 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 1 & -a_2/a_3 \\ 1 & 0 & -a_1/a_3 \\ 0 & 0 & 1/a_3 \end{bmatrix}$

$a_3 b_{31} = 0 \Rightarrow b_{31} = 0 \qquad a_3 b_{32} = 0 \Rightarrow b_{32} = 0 \qquad a_3 b_{33} = 1 \Rightarrow b_3 = \frac{1}{a_3}$

$b_{21} + \cancel{a_1 b_{31}} = 1 \qquad b_{22} + \cancel{a_1 b_{32}} = \emptyset \qquad b_{23} + a_1 b_{33} = 0$

$b_{21} = 1 \qquad b_{22} = 0 \qquad b_{23} + \frac{a_1}{a_3} = 0$

$b_{23} = -\frac{a_1}{a_3}$

$b_{11} + \cancel{a_2 b_{33}} = 0 \qquad b_{12} + \cancel{a_2 b_{32}} = 1 \qquad b_{13} + a_2 b_{33} = 0$

$b_{11} = 0 \qquad b_{12} = 1 \qquad b_{13} + \frac{a_2}{a_3} = 0$

$b_{13} = -\frac{a_2}{a_3}$

$$\boxed{A^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & -a_{n-1}/a_n \\ 0 & 0 & \cdots & 1 & 0 & -a_{n-2}/a_n \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & -a_2/a_n \\ 1 & 0 & \cdots & 0 & 0 & -a_1/a_n \\ 0 & 0 & \cdots & 0 & 0 & 1/a_n \end{bmatrix}}$$

$[1 \times 3][3\times 3] = [1 \times 3]$

3. (10 + 15 +10+ 5 pts) Consider matrix A,

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 8 & 2 \\ 2 & 2 & 6 \end{bmatrix} \qquad \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2} \tag{2}$$

(a) Is A positive definite ? Prove or disprove.

(b) Factorize A as $A = LL^T$ where L is lower triangular. Please show all steps.

(c) Solve $Ax = b$ for $b = [4\ \ 0\ \ 2]^T$ using forward and backward substitution. Solve $Lw=b$ Then $L^Tx = w$

(d) How many total flops does it take to solve for $Ax = b$ in the previous part (account for the cost of factorization).

(a) $[x_1\ \ x_2\ \ x_3]\begin{bmatrix} 2 & 2 & 2 \\ 2 & 8 & 2 \\ 2 & 2 & 6 \end{bmatrix} = [2x_1+2x_2+2x_3 \quad 2x_1+8x_2+2x_3 \quad 2x_1+2x_2+6x_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$2x_1^2 + 2x_1 2x_2 + 2x_1x_3 + 2x_1x_2 + 8x^2 + 2x_2x_3 + 2x_1x_3 + 2x_2x_3 + 6x_3^2 > 0$

$2x_1^2 + 8x_2^2 + 6x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 > 0$

$(x_1+x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2 \qquad (x_2+x_3)^2 = x_2^2 + 2x_2x_3^2 + x_3^2 \qquad (x_1+x_3)^2 = x_1^2 + 2x_1x_3 + x_3^2$

$(x_1+x_2)^2 + (x_2+x_3)^2 + (x_1+x_3)^2 + 6x_2^2 + 4x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 > 0$

$(x_1+x_2)^2 + 2(x_2+x_3^2) + (x_1+x_3)^2 + 5x_2^2 + 3x_3^2 > -2x_1x_2 - 2x_1x_3 \Rightarrow \boxed{A \text{ is P.D.}}$

A is symmetric, nonsingular, with (+) diagonal entries

−5

(b) $l_{11} = \sqrt{a_{11}} \quad L_{21} = \frac{1}{l_{11}}A_{21} \quad L_{22}L_{22}^T = A_{22} - L_{21}L_{21}^T$

1 $\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{2} & & \\ \sqrt{2} & & \end{bmatrix}$ 1 flop, 2 flops

$\begin{bmatrix} 8 & 2 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}[\sqrt{2}\ \ \sqrt{2}] = \begin{bmatrix} 8 & 2 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$ 4 flops, 4 flops

2 $\begin{bmatrix} \sqrt{6} & 0 \\ 0 & \end{bmatrix}$ 1 flop $\quad 4-(0)(0) = 4 \quad \sqrt{4} = 2$ 1 flop

$$\boxed{A = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{2} & \sqrt{6} & 0 \\ \sqrt{2} & 0 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

(c) $Lw = b$ $\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{2} & \sqrt{6} & 0 \\ \sqrt{2} & 0 & 2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

$\sqrt{2}w_1 = 4 \Rightarrow w_1 = \frac{4}{\sqrt{2}}$ (1 flop)

$\sqrt{2}\cdot\frac{4}{\sqrt{2}} + w_2\sqrt{6} = 0 \Rightarrow w_2\sqrt{6} = -4 \quad w_2 = \frac{-4}{\sqrt{6}}$ (3 flops)

$\sqrt{2}\cdot\frac{4}{\sqrt{2}} + w_3 2 = 2 \Rightarrow 2w_3 = -2 \quad w_3 = -1$ (3 flops)

$L^Tx = w$

$\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{2} \\ -4/\sqrt{6} \\ -1 \end{bmatrix}$ $\quad 4/\sqrt{2} \to 2\sqrt{2}$

$2x_3 = -1 \quad x_3 = -\frac{1}{2}$ (1 flop)

$\sqrt{6}x_2 = \frac{-4}{\sqrt{6}} \quad x_2 = -\frac{4}{6}$ (1 flop)

$\sqrt{2}x_1 - \frac{4\sqrt{2}}{6} - \frac{\sqrt{2}}{2} = 2\sqrt{2}$ (5 flops)

$\sqrt{2}x_1 - \frac{16\sqrt{2}}{24} - \frac{12\sqrt{2}}{24} = \frac{48\sqrt{2}}{24}$

$\sqrt{2}x_1 = \frac{48-28\sqrt{2}}{24} = x_1 = \frac{20}{24}$

$$\boxed{x = \begin{bmatrix} 5/6 \\ -2/3 \\ -1/2 \end{bmatrix}}$$

−2

(d) Cholesky 13 flops $\quad$ Foward 7 flops $\quad$ Backward 7 flops $\Rightarrow \boxed{27 \text{ flops}}$

−3

4. (10 pts) True or False.

10

Circling the correct answer is worth +2 points, circling an incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) Every triangular matrix is non-singular.

TRUE FALSE

(b) Schur complement of a positive definite matrix is positive definite.

TRUE FALSE

(c) All positive semi-definite matrices have strictly positive diagonal entries.

TRUE FALSE

(d) The norm $\|x\|$ of a unit vector x is zero.

TRUE FALSE

(e) Every matrix has a unique left inverse.

TRUE FALSE

5. (10 pts) For what values of a is the following matrix A positive definite ? For what values of a is the following matrix A positive semi-definite ? Please show all steps.

$$A = \begin{bmatrix} a & 0 & \cdots & 0 & 0 \\ 0 & a & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a & 0 \\ 0 & 0 & \cdots & 0 & a \end{bmatrix}$$

$$x^TA = [x_1 \quad x_2 \cdots x_{n-1} \quad x_n] \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & a & 0 \\ & & & 0 & a \end{bmatrix} = [ax_1 \quad ax_2 \cdots ax_{n-1} \quad ax_n]$$

$$x^TAx = [ax_1 \quad ax_2 \cdots ax_{n-1} \quad ax_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = ax_1^2 + ax_2^2 + \cdots + ax_{n-1}^2 + ax_n^2$$

$$x^TAx = a \underbrace{(x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^2)}_{\text{always positive b/c each term is squared}}$$

Positive definite

$x^TAx > 0$

$\boxed{P.D. \Rightarrow a > 0}$

Positive semidefinite

$x^TAx \geq 0$

$\boxed{P.S.D. \Rightarrow a \geq 0}$

6. (10 pts) Find the least norm solution for the following system of equations,

$$\hat{x} = A^T (AA^T)^{-1} b$$

$$\begin{cases} x_3 = 5 \\ x_1 + 2x_2 + 2x_4 = 0 \\ 4x_1 - x_2 - x_4 = 10. \end{cases}$$

10

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 2 \\ 4 & -1 & 0 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ 0 \\ 10 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 2 \\ 4 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+4+4 & 4-2-2 \\ 0 & 4-2-2 & 16+1+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 9 & 0 & 0 & 1 & 0 \\ 0 & 0 & 18 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{18} \end{array}\right] \qquad (AA^T)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{18} \end{bmatrix}$$

$$(AA^T)^{-1} b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{18} \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ \frac{10}{18} \end{bmatrix}$$

$$A^T (AA^T)^{-1} b = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ \frac{5}{9} \end{bmatrix} = \begin{bmatrix} 20/9 \\ -5/9 \\ 5 \\ -5/9 \end{bmatrix}$$

$$\boxed{\hat{x} = \begin{bmatrix} 20/9 \\ -5/9 \\ 5 \\ -5/9 \end{bmatrix}}$$