ECE102, Systems and Signals
UCLA Summer 2020
Midterm Exam
07/21/2020-07/22/2020
Time Limit: 18 hours

Name: ______________________

(a) This exam contains 7 pages (including this cover page) and 6 problems. Total of points is 100.

(b) You must scan and submit your solutions in a single file (preferably in PDF format) via the CCLE portal by **Wednesday 07/22/20, 12:00pm (noon) at the latest**. Please note that the portal will shut down at 12pm sharp.

(c) Please make sure to write your full name on your papers and also **include your name as part of your file name**.

(d) Please fully justify your answers and clearly show ALL the intermediate steps and calculations in all your solutions. And, when appropriate, box your final answer.

(e) Open books and notes.

(f) Feel free to use any calculators and computers, including MATLAB/Python, if you like. But, except when specifically asked for, using them is not advisable and will likely waste your time without helping much for the solutions. And, regardless, please remember that all your intermediate steps and calculations must be included in your answers.

(g) And finally, no collaboration please. You are expected to work on the solutions individually. Unreasonably similar write-ups and calculation steps would be heavily penalized on all parties suspected of collaboration.

(h) **Good Luck...**

Grade Table (for instructor use only)

Question	Points	Score
1	15	
2	18	
3	18	
4	12	
5	19	
6	18	
Total:	100	

1. (15 points) **Review**
Carefully read each statement below and identify it as *True* or *False* and, for each answer, **explain your reasoning briefly in 1-2 sentences.**

   1. A 100KHz tone (i.e., sinusoid) is input to an LTI system. The frequency spectrum of the output signal may include components at integer multiples of 100KHz.
   True or False? Why?
   2. The phase of the Fourier transform of a real-valued signal with odd symmetry will always be ± 90 deg.
   True or False? Why?
   3. The input-output of an LTI system is fully characterized once we know its impulse response.
   True or False? Why?
   4. The impulse response sequence of a discrete-time accumulator is absolutely summable.
   True or False? Why?
   5. The output of a *linear time-varying* system to a periodic input signal will always be periodic with the same fundamental period as the input.
   True or False? Why?
   6. The signal $a_1 \sin(\omega_1 t) + a_2 \sin(\omega_2 t)$ is always periodic for any values of frequencies ω_1 and ω_2.
   True or False? Why?
   7. A periodic signal with even symmetry (i.e., $x(t) = x(-t)$) will always be even harmonic (i.e., its Fourier series coefficients $a_k = 0$ for k odd).
   True or False? Why?
   8. An LTI system must be BIBO stable for us to be able to obtain and characterize its frequency response.
   True or False? Why?
   9. Any signal that has finite energy will be absolutely integrable too.
   True or False? Why?
   10. The *magnitude* frequency spectrum of any real-valued signal will always have even symmetry with respect to frequency.
   True or False? Why?

Problem 1

1) False. An LTI system can never generate new frequencies at its output. So, with a 100KHz tone at its input, its steady-state output with be a 100KHz tone whose amplitude is scaled by the magnitude of the freq. respone and whose phase is shifted by the phase of the freq. response at 100KHz.

2) True. The FT of a real and odd signal is odd and purely imaginary. Thus its phase is $\pm 90°$.

3) True. Thanks to superposition and hence the convolution integral.

4) False. A discrete_time accumulator is unstable and therefore its impulse response is not absolutely summable.

5) False. If the system is not time-Invariant, it may not preserve the periodicity of its input.

6) False. Only true if $\frac{\omega_1}{\omega_2}$ is a rational number.

7) False. A signal with even symmetry may not necessarily be even harmonic. Think of $\cos(t)$ as a simple example.

8) True. The impulse response of an unstable system will grow unbounded and its FT integral would not converge.

9) False. $\int_{-\infty}^{\infty} |h(t)|^2 dt < \infty$ does not necessarily imply $\int_{-\infty}^{\infty} |h(t)| dt < \infty$.

An example is the sinc function which does have finite energy but it is not absolutely integrable.

The reverse of this is true though, i.e. If a signal is absolutely integrable, it will have finite energy.

10) True. For any real-valued $x(t)$, we have:

$$|X(f)| = |X(-f)|$$
$$\measuredangle X(f) = -\measuredangle X(-f)$$

2. (18 points) The input-output relationship for an LTI system is given as follows:

$$y(t) = \int_{-\infty}^{t} x(\tau - 1)(t - \tau)e^{-2(t-\tau)}d\tau \qquad (1)$$

(a) (5 points) Find the Impulse Response $h(t)$ for this system.

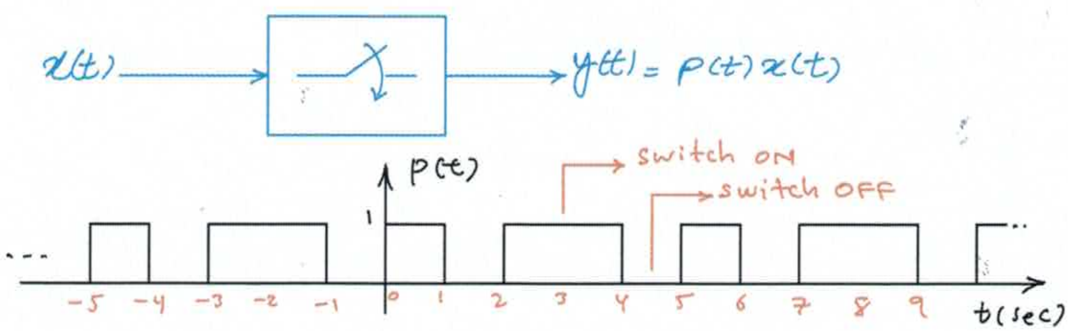
(b) (4 points) Find the Step Response $g(t)$ for this system.

(c) (4 points) Find the Frequency Response $H(j\omega)$ for this system. (Feel free to use the CTFT table along with the CTFT properties).

(d) (5 points) Obtain the steady-state response of this system to the following input signal:

$$x(t) = 10 + 100\cos(2t + \frac{\pi}{3})$$

3. (18 points) Consider the following switching system, where the system output can be modeled as the input signal multiplied by the specific pulse train shown below:



(a) (1 point) Is this system linear? Please explain.

(b) (1 point) Is this system causal? Please explain.

(c) (1 point) Is this system static (memoryless) or dynamic? Please explain.

(d) (2 points) Is this system time-invariant or time-varying? Please explain.

(e) (5 points) Find the *Fourier Series* coefficients for the given pulse train $p(t)$.

(f) (3 points) Using MATLAB (or Python), plot the magnitude and phase spectra for $p(t)$ for the first 10 harmonics (e.g., in MATLAB, set `k=-10:10` and use `stem` command, e.g., `stem (k,abs(ak))` and `stem (k,angle(ak)*180/pi)`.

(g) (5 points) Assume the input signal to this system is a complex exponential:

$$x(t) = e^{j\frac{4\pi}{5}t}$$

Obtain the *Fourier Transform* of the output signal $y(t)$ and plot its magnitude spectrum $|Y(f)|$ with respect to the cyclic frequency f over the range of $f = [-1, 1]$ Hz. Note that you need to plot this carefully by hand, though you can take advantage of the magnitude line spectra you plotted in Part (f).

problem 2

The input-output relationship for an LTI system is given as:

$$y(t) = \int_{-\infty}^{t} x(\tau-1)(t-\tau)e^{-2(t-\tau)}\,d\tau$$

a) Find its impulse response $h(t)$.

$$y(t) = \int_{-\infty}^{t} x(\tau-1)(t-\tau)e^{-2(t-\tau)}\,d\tau$$

$\tau' \triangleq \tau - 1$

$$= \int_{-\infty}^{t-1} x(\tau')(t-1-\tau')e^{-2(t-1-\tau')}\,d\tau'$$

$$= \int_{-\infty}^{\infty} x(\tau')(t-1-\tau')e^{-2(t-1-\tau')}u(t-1-\tau')\,d\tau'$$

$$= \int_{-\infty}^{\infty} x(\tau')h(t-\tau')\,d\tau'$$

$$\Rightarrow h(t) = (t-1)e^{-2(t-1)}u(t-1)$$

b) Find its step response $g(t)$.

$$g(t) = \int_{-\infty}^{t} h(\tau)\,d\tau = \int_{-\infty}^{t} (\tau-1)e^{-2(\tau-1)}u(\tau-1)\,d\tau$$

$$= \int_{1}^{t} (\tau-1)e^{-2(\tau-1)}\,d\tau$$

$$\tau' \triangleq \tau - 1 \longrightarrow \; = \int_0^{t-1} \tau' e^{-2\tau'} d\tau'$$

Integration by part : $\int u\,dv = uv - \int v\,du$

$u \triangleq \tau'$, $dv \triangleq e^{-2\tau'} d\tau' \longrightarrow v = -\frac{1}{2} e^{-2\tau'}$

$$\Rightarrow g(t) = -\frac{\tau'}{2} e^{-2\tau'} \Big|_0^{t-1} + \frac{1}{2}\int_0^{t-1} e^{-2\tau'} d\tau'$$

$$= -\frac{\tau'}{2} e^{-2\tau'} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) e^{-2\tau'} \Big|_0^{t-1}$$

$$= -\frac{e^{-2\tau'}}{2}\left(\tau' + \frac{1}{2}\right)\Big|_0^{t-1} = -\frac{e^{-2(t-1)}}{2}\left(t-1+\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) \quad ; \text{for } t>1$$

$$\Rightarrow$$

$$g(t) = \left[\frac{e^{-2(t-1)}}{2}\left(\frac{1}{2} - t\right) + \frac{1}{4}\right] u(t-1)$$

c) Find its frequency response $H(j\omega)$.

(Feel free to use the CTFT table along with CTFT properties)

$$t e^{-at} u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{(a+j\omega)^2}$$

$$\Rightarrow H(j\omega) = \frac{1}{(2+j\omega)^2} \cdot \underbrace{e^{-j\omega}}_{\text{due to 1 sec delay}}$$

d) What is the steady-state response of this system to the following input: $x(t) = 10 + 100\cos(2t + \frac{\pi}{3})$

$$y_{ss}(t) = 10 \cdot H(j0) + 100\,|H(j2)| \cos\left(2t + \frac{\pi}{3} + \measuredangle H(j2)\right)$$

$$H(j0) = 0.25$$

$$H(j2) = \frac{1}{(2+j2)^2} e^{-j2} = \frac{1}{4(1+j)^2} \cdot e^{-j2}$$

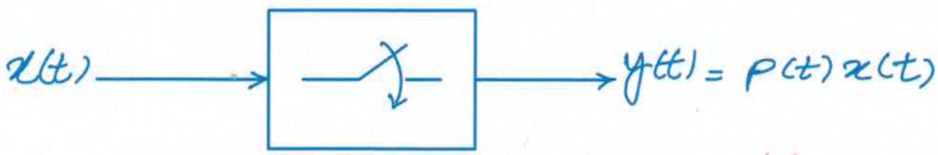
$$= \frac{e^{-j2}}{4} \cdot \frac{(1-j)^2}{(1+1)^2} = \frac{e^{-j2}}{4} \cdot \frac{-2j}{4}$$

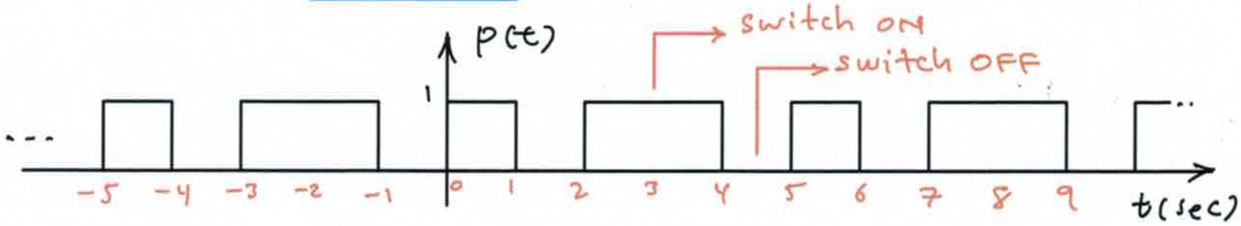
$$= \frac{1}{8} e^{-j(2+\frac{\pi}{2})}$$

$$\Rightarrow y_{ss}(t) = 2.5 + 12.5\cos\left(2t + \frac{\pi}{3} + 2 + \frac{\pi}{2}\right)$$

problem 3

Consider the following switching system. The system output can be modeled as the input multiplied by a specific pulse train as shown below:





a) Is this system linear?

– yes, because superposition holds true:

$$p(t)\left[a_1 x_1(t) + a_2 x_2(t)\right] = a_1 p(t) x_1(t) + a_2 p(t) x_2(t)$$

b) Is this system causal?

– yes, the output at time t only depends on the input at time t and no future input values.

c) Is this system static (memoryless) or dynamic?

– Static, because the output at t only depends on the input at that same time instant.

d) Is this system time-invariant?

– No, it is time-varying, because:

$$x_2(t) \triangleq x(t-t_0) \rightarrow y_2(t) = p(t)x_2(t) = p(t)x(t-t_0)$$

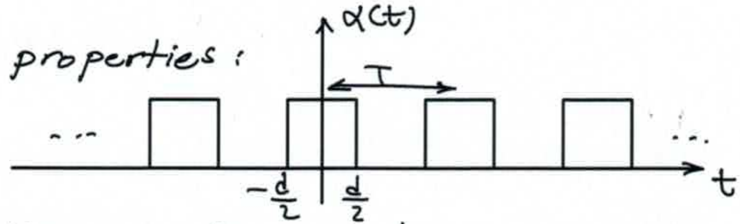
$$\neq p(t-t_0)x(t-t_0) = y(t-t_0)$$

e) Find the Fourier Series coefficients of $p(t)$.

– $p(t)$ is clearly periodic with the fundamental period of $T=5$ sec. We can find its FS coefficients either directly or by using FS properties along with our known FS coefficients for a regular pulse train.

I) Using FS properties:

Recall:



$$\alpha(t) \triangleq \sum_k \Pi\left(\frac{t-kT}{d}\right) \xleftrightarrow{FS} a_k = \frac{d}{T}\,\text{Sinc}\left(\frac{kd}{T}\right) \quad (I)$$

and notice that we can write:

$$p(t) = p_1(t) + p_2(t)$$

where $p_1(t)$ and $p_2(t)$ are two pulse trains, both with period 5 sec.

$p_1(t) = \alpha(t - \frac{1}{2})$ with $d=1$ and $T=5$

$p_2(t) = \alpha(t-3)$ with $d=2$ and $T=5$

Therfore using (I) above along with the time delay property of FS we have:

$$p_1(t) \xleftrightarrow{FS} a_{p_{1_k}} = \frac{1}{5}\,\text{Sinc}\left(\frac{k}{5}\right) \cdot e^{-jk\frac{2\pi}{5}\cdot\frac{1}{2}}$$

$$p_2(t) \xleftrightarrow{FS} a_{p_{2_k}} = \frac{2}{5}\,\text{Sinc}\left(\frac{2k}{5}\right) \cdot e^{-jk\frac{2\pi}{5}\cdot 3}$$

And using linearity property of FS, we have:

$$p(t) \xleftrightarrow{FS} a_{p_{1k}} + a_{p_{2k}} = \frac{1}{5}\,\text{Sinc}\left(\frac{k}{5}\right)e^{-j\frac{\pi k}{5}} + \frac{2}{5}\,\text{Sinc}\left(\frac{2k}{5}\right)e^{-j\frac{6\pi k}{5}}$$

II) Direct Method:

$$a_{p_k} = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{5}\int_0^1 e^{-jk\omega_0 t}\,dt + \frac{1}{5}\int_2^4 e^{-jk\omega_0 t}\,dt$$

$$= -\frac{1}{j5k\omega_0}\left[e^{-jk\omega_0 t}\right]_0^1 - \frac{1}{j5k\omega_0}\left[e^{-jk\omega_0 t}\right]_2^4$$

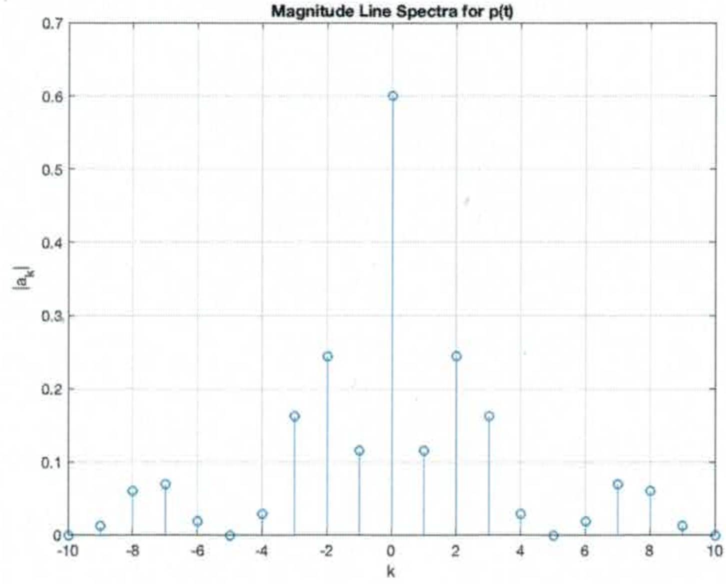
$$= -\frac{1}{j5k\cancel{\omega_0}_{\frac{2\pi}{5}}}\left[e^{-jk\omega_0} - 1\right] - \frac{1}{j5k\omega_0}\left[e^{-j4k\omega_0} - e^{-j2k\omega_0}\right]$$

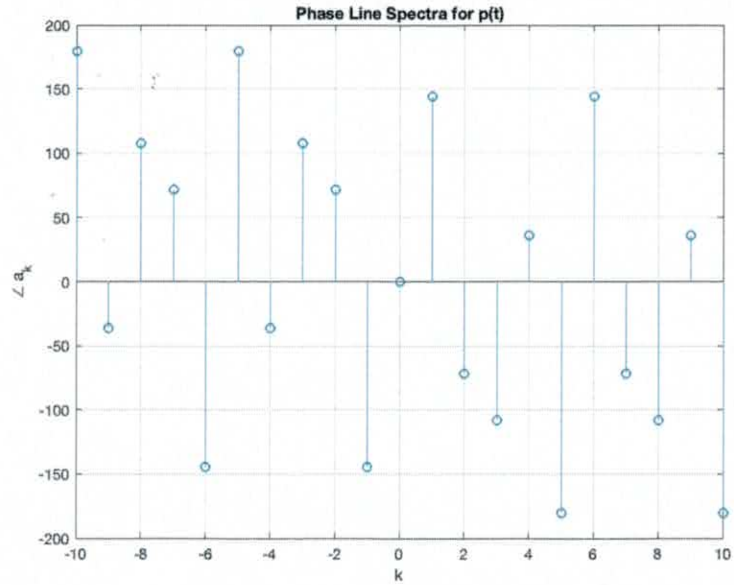
$$= -\frac{1}{j2\pi k}\cdot e^{-j\frac{\pi k}{5}}\left[\overbrace{e^{-j\frac{\pi k}{5}} - e^{j\frac{\pi k}{5}}}^{-2j\,\text{Sin}\left(\frac{\pi k}{5}\right)}\right]$$

$$- \frac{1}{j2\pi k}\,e^{-j\frac{6\pi k}{5}}\left[\underbrace{e^{-j\frac{2\pi k}{5}} - e^{j\frac{2\pi k}{5}}}_{-2j\,\sin\left(\frac{2\pi k}{5}\right)}\right]$$

$$\Rightarrow a_{p_k} = \frac{1}{5}\,\text{Sinc}\left(\frac{k}{5}\right)e^{-j\frac{\pi k}{5}} + \frac{2}{5}\,\text{Sinc}\left(\frac{2k}{5}\right)e^{-j\frac{6\pi k}{5}}$$

f) Using MATLAB, Plot the magnitude and the phase spectra for p(t) for the first 10 harmonics (i.e., for $k=-10:10$)





g) Now, assume the input signal is a complex exponential:

$$x(t) = e^{j\frac{4\pi}{5}t}$$

Obtain the Fourier Transform of $y(t)$

and plot its magnitude spectrum $|Y(f)|$ with respect to the cyclic frequency f and over the range of $[-1, 1]$ Hz.
Note that you need to plot this carfully by hand, though you can use the magnitude line spectra plot in part f.

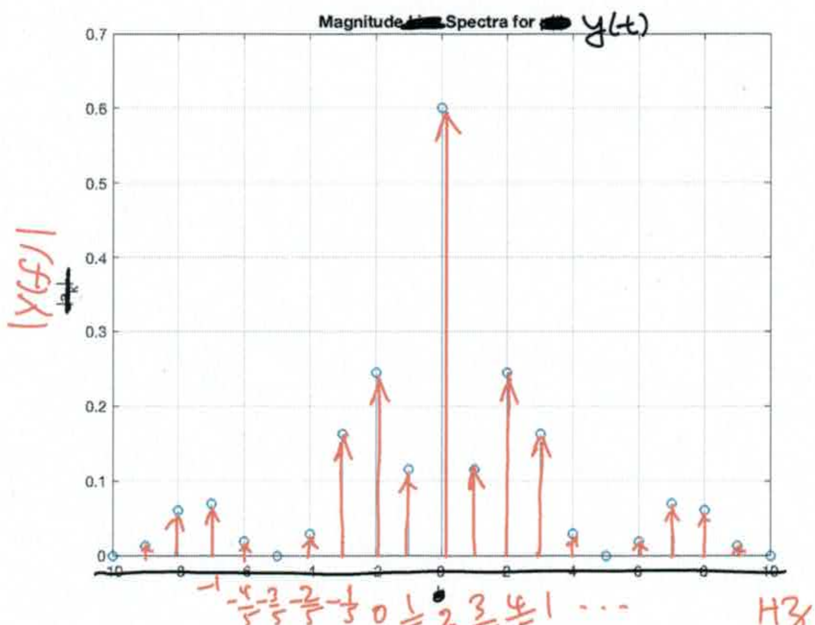
We have: $x(t) = e^{j\frac{4\pi}{5}t} \longrightarrow X(f) = \delta(f - \frac{2}{5})$

$$P(f) = \sum_k a_{p_k} \delta(f - \frac{k}{5})$$

where a_{p_k} was obtained in part e and

$|a_{p_k}|$ was plotted in part f.

From the frequency shift or modulation property: $Y(f) = P(f - \frac{2}{5}) = \sum_k a_k \delta(f - \underline{\frac{k+2}{5}})$



4. (12 points) (a) (3 points) In class, we proved:

$$\mathscr{F}\{\mathrm{sgn}(t)\} = \frac{1}{j\pi f}$$

Using a similar approach, along with CTFT properties, show that:

$$\mathscr{F}^{-1}\{\mathrm{sgn}(f)\} = \frac{j}{\pi t}$$

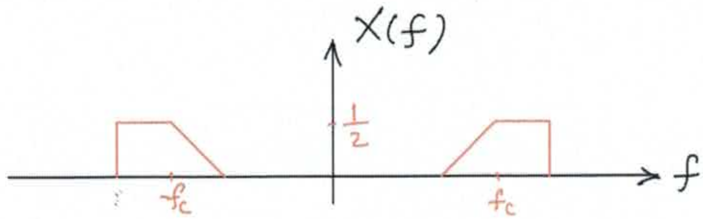
where $\mathrm{sgn}(f)$ is the signum or sign function in frequency.

(b) (3 points) Using the result in Part (a), show that:

$$\mathscr{F}^{-1}\{U(f)\} = \frac{1}{2}\delta(t) + \frac{j}{2\pi t}$$

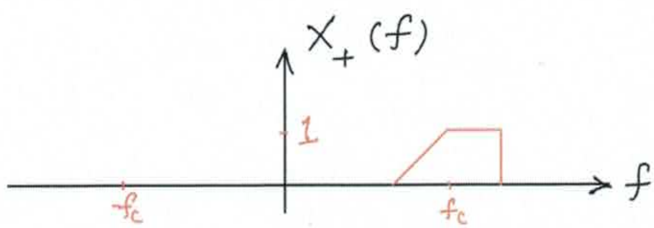
where $U(f)$ is the unit step function in frequency.

(c) (6 points) Now, consider a signal $x(t)$ with the following frequency spectrum:



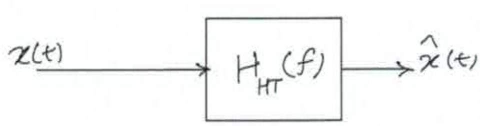
And define the signal $x_+(t)$ as a new signal that only has the positive frequency components of $x(t)$, i.e.,

$$X_+(f) = 2X(f)U(f)$$



Show that $x_+(t)$ can be written as $x_+(t) = x(t) + j\hat{x}(t)$ where $\hat{x}(t)$ is the output of the following filter:

$$H_{HT}(f) = -j\,\mathrm{sgn}(f)$$



The filter $H_{HT}(f)$ is called the *Hilbert Transform* filter and it has many applications in signal processing and communications. It is essentially an *all-pass* filter which introduces a 90 deg phase shift.

Problem 4

a) In class, we proved: $\mathcal{F}\{sgn(t)\} = \frac{1}{j\pi f}$

Using CTFT properties, prove that:

$$\mathcal{F}^{-1}\{sgn(f)\} = \frac{j}{\pi t} \quad , \quad sgn(f) = \begin{cases} -1 & f<0 \\ 0 & f=0 \\ 1 & f>0 \end{cases}$$

$$\frac{d}{df} sgn(f) = 2\delta(f) \Rightarrow \mathcal{F}^{-1}\left\{\frac{d}{df} sgn(f)\right\} = 2 \quad (I)$$

From "Derivative in Freq" property:

$$\mathcal{F}^{-1}\left\{\frac{d}{df} sgn(f)\right\} = -j2\pi t\, \mathcal{F}^{-1}\{sgn(f)\} \quad (II)$$

From (I) and (II), we have:

$$\mathcal{F}^{-1}\{sgn(f)\} = -\frac{1}{j\pi t} = \frac{j}{\pi t}$$

b) Using the result in part a), show that:

$$\mathcal{F}^{-1}\{U(f)\} = \frac{1}{2}\delta(t) + \frac{j}{2\pi t}$$

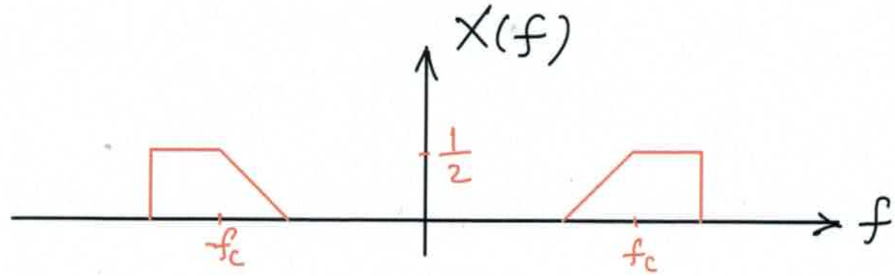
Where $U(f)$ is a unit step function in frequency.

$$U(f) = \frac{1}{2} + \frac{1}{2} sgn(f)$$

Using the result in part a):

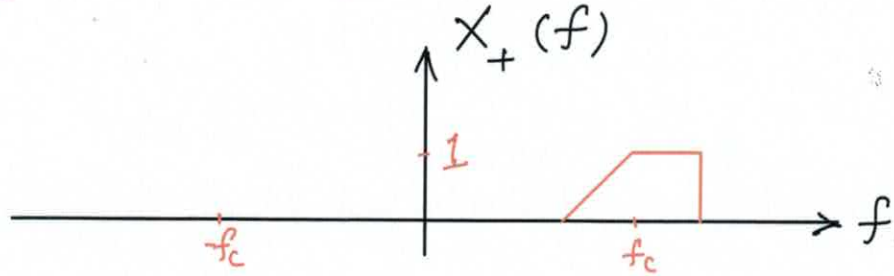
$$\mathcal{F}^{-1}\{U(f)\} = \frac{1}{2}\delta(t) + \frac{j}{2\pi t}$$

C) Consider a signal $x(t)$ with the following spectrum:



and define $x_+(t)$ as a signal that only has the positive frequency components of $x(t)$, i.e.:

$$X_+(f) = 2X(f)\,U(f)$$



Show that $x_+(t) = x(t) + j\hat{x}(t)$

where $\hat{x}(t)$ is the output of the following filter, known as "Hilbert Transform" filter:

$$H_{HT}(f) = -j\,\mathrm{sgn}(f)$$

$$x(t) \longrightarrow \boxed{H_{HT}(f)} \longrightarrow \hat{x}(t)$$

$$X_+(f) = 2X(f)\,U(f)$$

$$\rightarrow x_+(t) = \mathcal{F}^{-1}\{X_+(f)\}$$

$$= 2x(t) * \mathcal{F}^{-1}\{U(f)\}$$

$$= 2x(t) * \left(\frac{1}{2}\delta(t) + \frac{j}{2\pi t}\right)$$

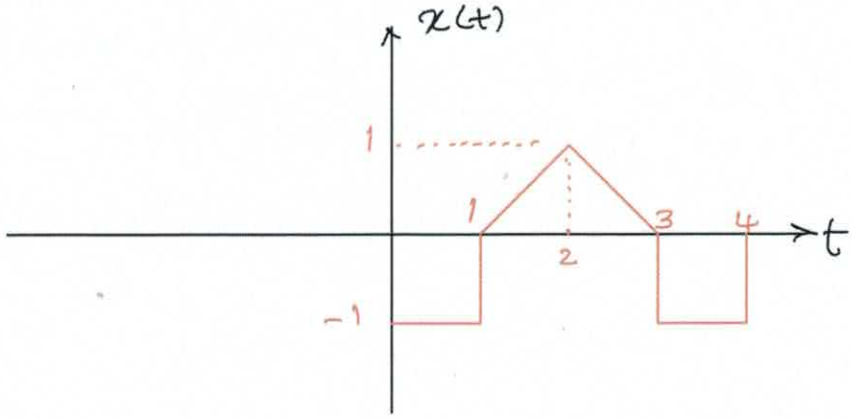
$$= x(t) * \delta(t) + x(t) * \frac{j}{\pi t}$$

$$= x(t) + j\,\underbrace{x(t) * \frac{1}{\pi t}}_{\hat{x}(t)}$$

$\rightarrow$ $\hat{x}(t)$ is the output of a filter with impulse response: $h_{HT}(t) = \dfrac{1}{\pi t}$

$$\rightarrow H_{HT}(f) = \mathcal{F}\left\{\frac{1}{\pi t}\right\} = \frac{1}{j}\,\mathrm{sgn}(f) = -j\,\mathrm{sgn}(f)$$

5. (19 points) Consider the following signal:



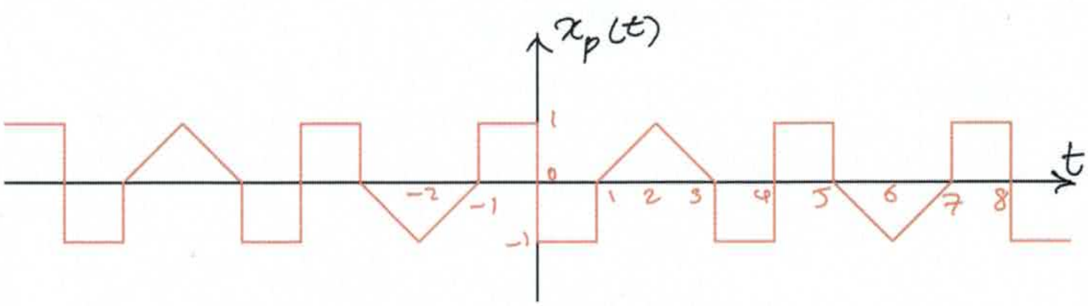
(a) (5 points) Find its Fourier Transform $X(f)$. Feel free to take advantage of the examples we have done in the class, along with the CTFT properties.

(b) (4 points) Using MATLAB (or Python), plot the magnitude and phase spectra of $x(t)$ versus the cyclic frequency f in the range of [-3,3] Hz. Use linear scales for both horizontal and vertical axes. (e.g., set your frequency vector to, say, `f=-3.0:0.01:3.0`, evaluate X from Part (a) over `f`, and plot `abs(X)` and `angle(X)`. In evaluating X, you can use `exp` function for exponential if needed and note that for element-wise multiplication of two vectors, you can simply add a dot, e.g., `a.*b` for multiplying elements of `a` and `b`).

(c) (2 points) What is the *DC Component* of $x(t)$?

(d) (2 points) Evaluate $\int_{-\infty}^{\infty} |X(f)|^2 df$.

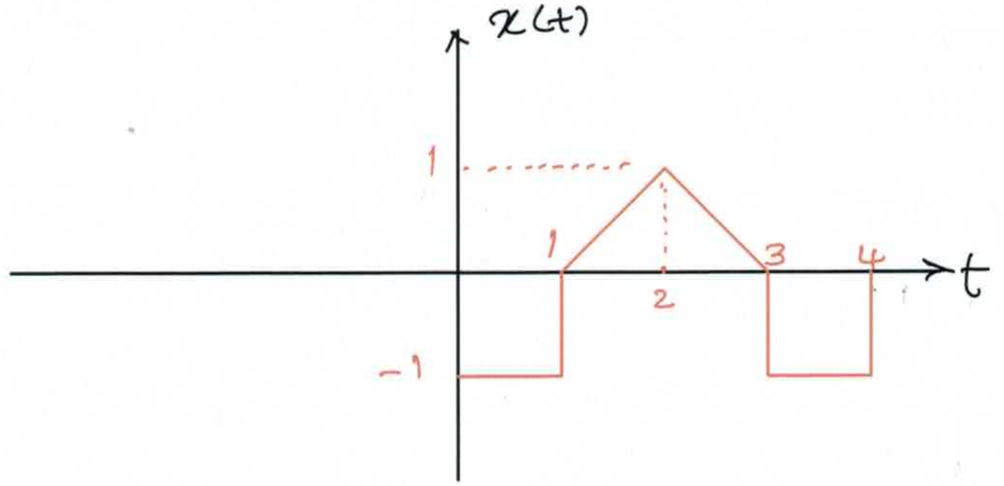
(e) (6 points) Now, consider the periodic signal $x_p(t)$ as shown below:



Obtain the Fourier Series coefficients a_k for $x_p(t)$, by writing one of its periods in terms of our original signal $x(t)$ and then using the $X(f)$ that you already found in Part (a).

problem 5

Consider the following signal:



a) Find its Fourier Transform $X(f)$.

We can find $X(f)$ either directly or using CTFT properties.

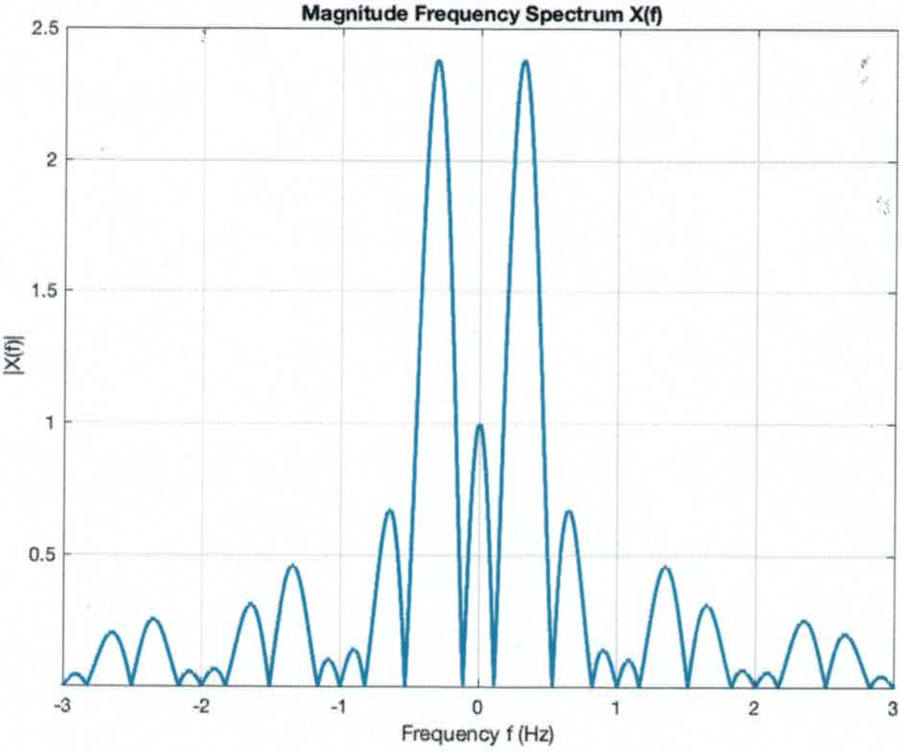
$$x(t) = -\Pi\left(\frac{t-0.5}{1}\right) + \Lambda\left(\frac{t-2}{1}\right) - \Pi\left(\frac{t-3.5}{1}\right)$$

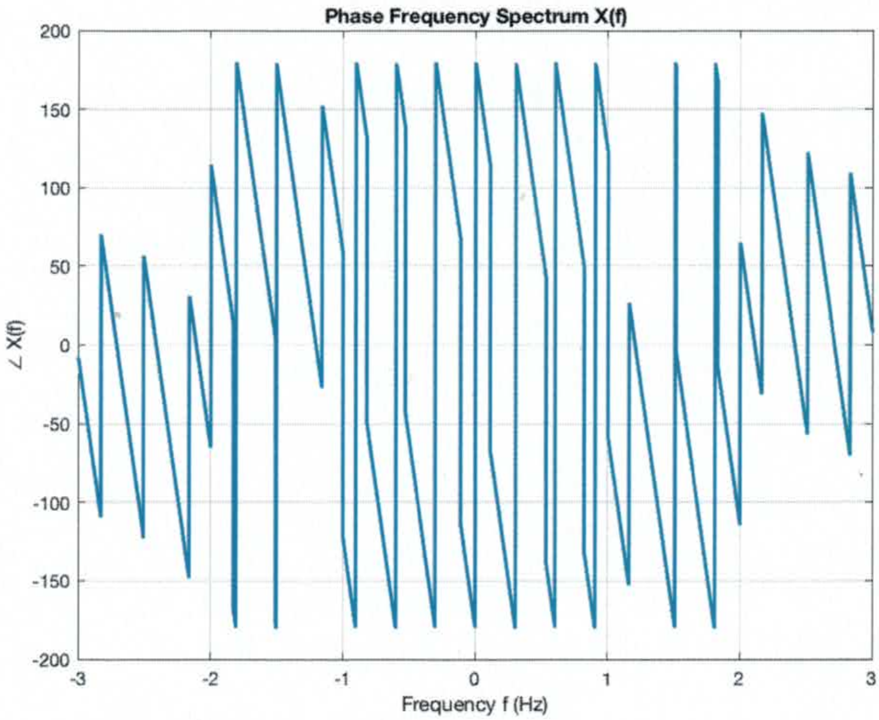
$$\Rightarrow X(f) = -\operatorname{sinc}(f)\, e^{-j2\pi f \cdot \frac{1}{2}}$$

$$+ \operatorname{sinc}^2(f)\, e^{-j2\pi f \cdot 2} - \operatorname{sinc}(f)\, e^{-j2\pi f \cdot \frac{7}{2}}$$

$$\Rightarrow X(f) = e^{-j\pi f}\,\text{sinc}(f)\left(\text{sinc}(f)\cdot e^{-j3\pi f} - 1 - e^{-j6\pi f}\right)$$

b) Using MATLAB, plot its magnitude and phase spectrum versus frequency f in the range of [-3,3] Hz. Use linear scales for both horizontal and vertical axes.





c) What is the DC component of this signal?

$X(0) = -1$ which can be seen both from $X(f)\big|_{f=0}$ and also by calculating the area under $x(t)$.

$$X(0) = \int_{-\infty}^{\infty} x(t)\,dt = -1$$

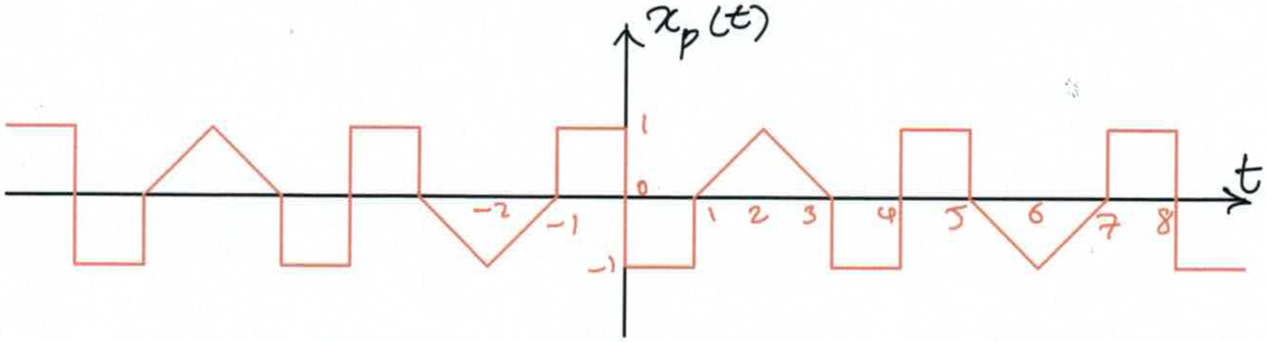
d) Evaluate $\int_{-\infty}^{\infty} |X(f)|^2 df$.

Recall from Parseval's energy relation:

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= 3$$

e) Now, consider the following periodic signal.



Find the Fourier Series coefficients for $x_p(t)$, by writing one of its periods in terms of $x(t)$ and then using the $X(f)$ you found in part a).

$$x_2(t) \triangleq x(t) - x(t-4)$$

Then $x_p(t)$ is clearly the "periodic extension" of $x_2(t)$.

we have: $X_2(f) = X(f)\left(1 - e^{-j2\pi f \cdot 4}\right)$

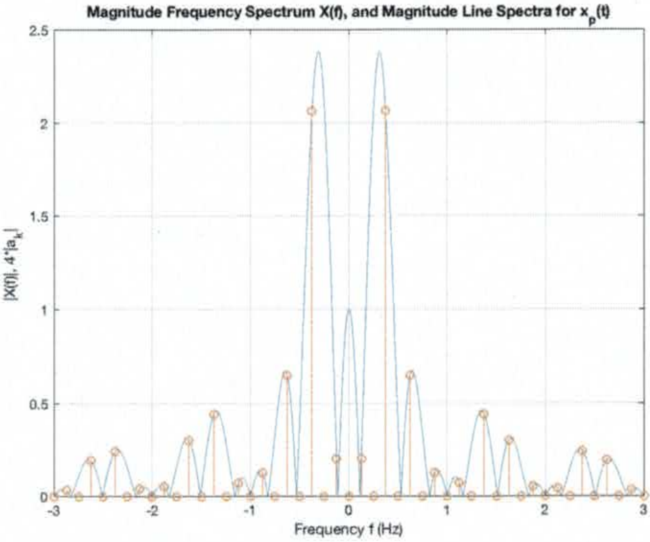
Therefore (see the begining of Lecture6):

$$a_k = \frac{1}{T} X_2(f)\Big|_{f=\frac{k}{T}}$$

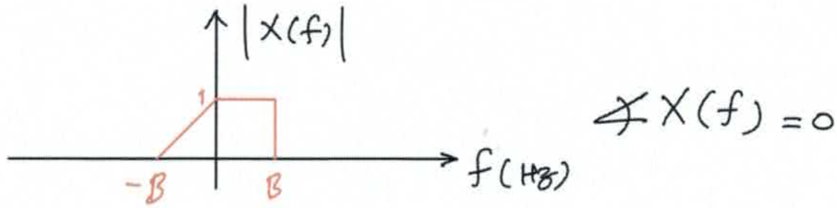
$$\Rightarrow a_k = \frac{1}{8} X\left(\frac{k}{8}\right)\left(1 - e^{-j8\pi \cdot \frac{k}{8}}\right)$$

$$\Rightarrow a_k = \frac{1}{8} X\left(\frac{k}{8}\right) \cdot \underbrace{\left(1 - e^{-j\pi k}\right)}_{= \begin{cases} 0, & k \text{ even} \\ 2, & k \text{ odd} \end{cases}}$$

$$\Rightarrow a_k = \begin{cases} \frac{1}{4} X\left(\frac{k}{8}\right) & ; k \text{ odd} \\ 0 & ; k \text{ even} \end{cases}$$

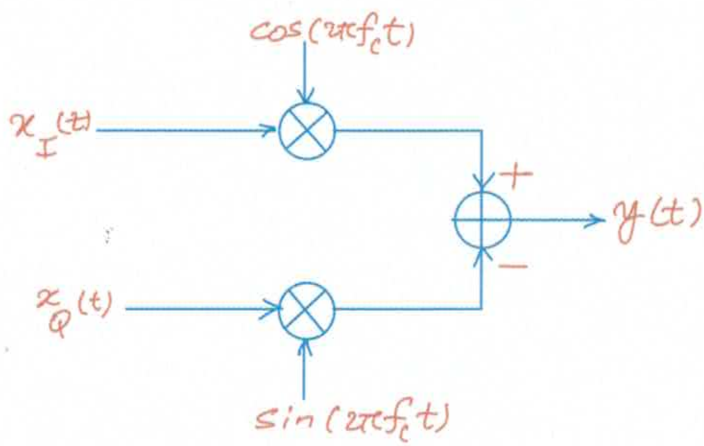


6. (18 points) Consider a complex-valued signal $x(t) = x_I(t) + jx_Q(t)$ with its frequency spectrum $X(f) = \mathscr{F}\{x(t)\}$ shown below:



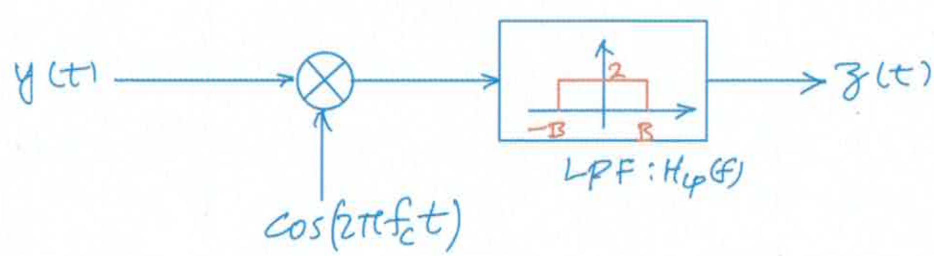
(a) (2 points) Obtain and plot $\mathscr{F}\{x^*(t)\}$ where $x^*(t)$ is the complex-conjugate of $x(t)$.

(b) (6 points) Assume $x(t)$ is our information-carrying signal. The modulated signal $y(t)$ is then formed as shown below ($f_c > B$):

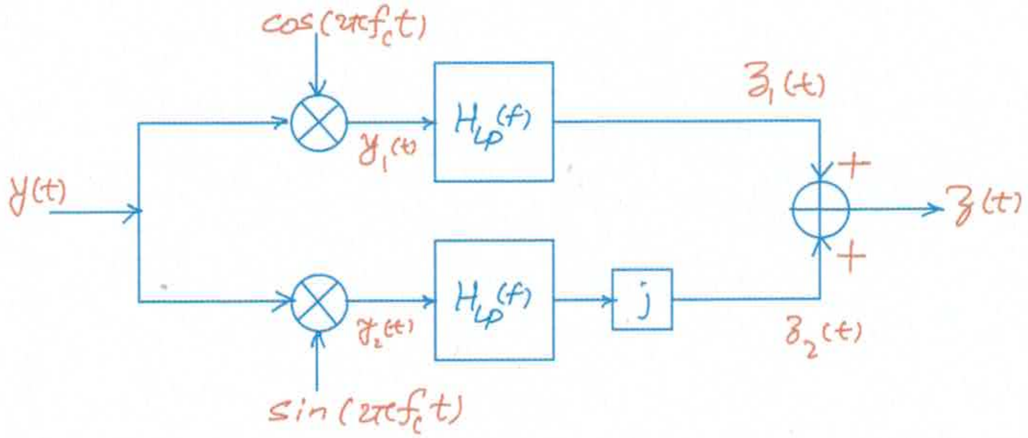


Obtain the CTFT of $y(t)$, i.e., $Y(f)$, in terms of $X(f)$ and plot $|Y(f)|$. This is called an In-phase/Quadrature or *I/Q Modulator* (*Hint:* Notice that $y(t) = \Re\{x(t)e^{j2\pi f_c t}\}$ and for any complex number a, we have $\Re\{a\} = \frac{a+a^*}{2}$).

(c) (4 points) Now, assume our communication channel was perfect and we received $y(t)$, as is, at the input to our receiver. Can the receiver shown below be used to recover our original signal $x(t)$? Why? Please explain your answer by obtaining and plotting the CTFT of the output, i.e., $|Z(f)|$.



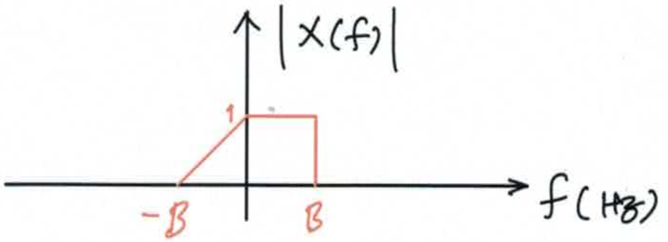
(d) (6 points) Now, assume instead we use the following receiver:



where $H_{LP}(t)$ is the same ideal Low-Pass Filter with cut-off at B Hz as shown in Part (c). Obtain and plot $Z(f) = \mathscr{F}\{z(t)\}$, and prove that $z(t)$ will indeed be equal to $x(t)$. This is called an *I/Q Demodulator.*

Problem 6:

Consider a complex-valued signal $x(t) = x_I(t) + j\,x_Q(t)$ with its frequency spectrum shown below:

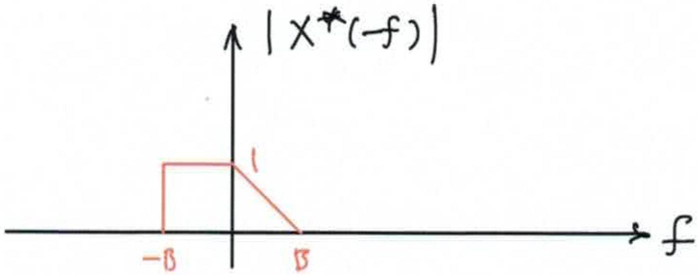


$\measuredangle X(f) = 0$

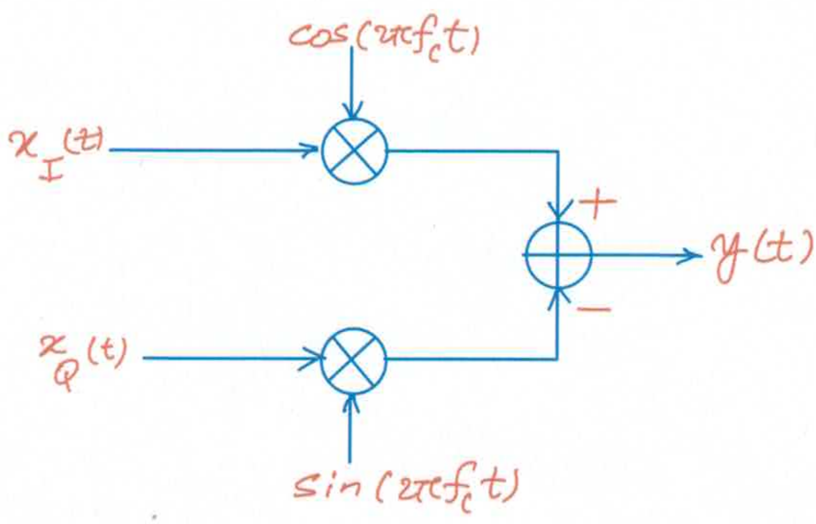
a) What is $\mathcal{F}\{x^*(t)\}$ ($x^*(t)$: complex conjugate of $x(t)$) in terms of $X(f)$? Plot the associated freq. spectrum.

From complex conjugate property of FT:

$$\mathcal{F}\{x^*(t)\} = X^*(-f) = X(-f) \quad (\text{since } \measuredangle X(f) = 0)$$



b) The signal $y(t)$ is formed as shown below ($f_c > B$):



Obtain the CTFT of $y(t)$, i.e. $Y(f)$, in term of $X(f)$, and plot $|Y(f)|$.

$$y(t) = x_I(t)\cos 2\pi f_c t - x_Q(t)\sin 2\pi f_c t$$
$$= \mathrm{Re}\left\{ (x_I(t) + j x_Q(t))\, e^{j2\pi f_c t} \right\}$$
$$= \frac{1}{2} x(t) e^{j2\pi f_c t} + \frac{1}{2} x^*(t) e^{-j2\pi f_c t}$$

From complex conjugate and freq shift properties of CTFT, we have:

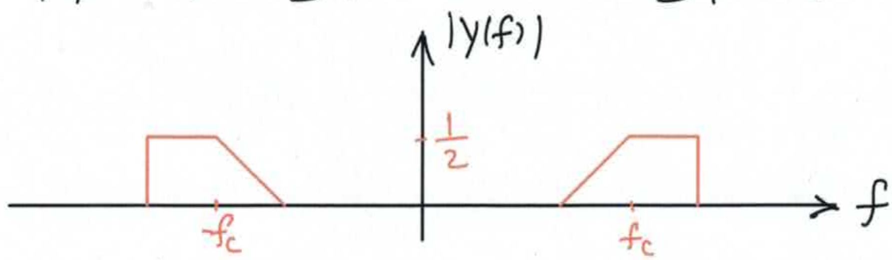
$$x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-f)$$
$$x(t) e^{j2\pi f_c t} \xleftrightarrow{\mathcal{F}} X(f-f_c)$$

Therefore:

$$Y(f) = \frac{1}{2} X(f-f_c) + \frac{1}{2} X^*(-(f+f_c))$$

Since $f_c > B$ in this case, and hence there is no overlap in freq. between $X(f-f_c)$ and $X^*(-f-f_c)$, we can write:

$$|Y(f)| = \frac{1}{2}|X(f-f_c)| + \frac{1}{2}|X(-f-f_c)|$$

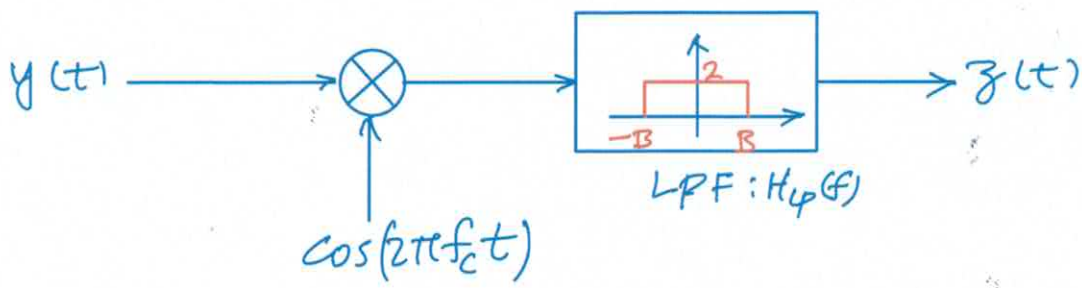


Notice the even symmetry in $Y(f)$ given that $y(t)$ is now a real-valued signal.

c) Assume $x(t)$ is our information-carrying signal and assume $y(t)$ is the signal received at the input to our receiver.

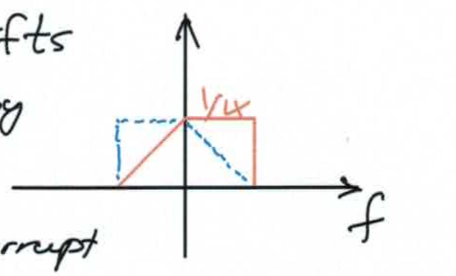
Can the receiver shown below be used to recover our original signal $x(t)$?

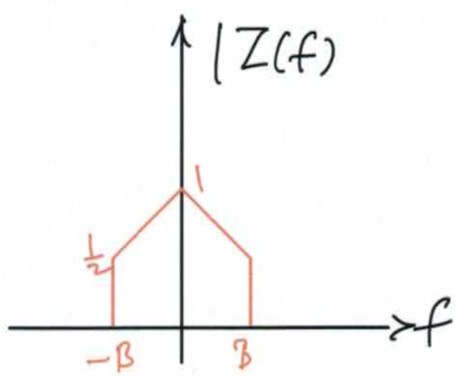
Explain your answer by plotting $|Z(f)|$.



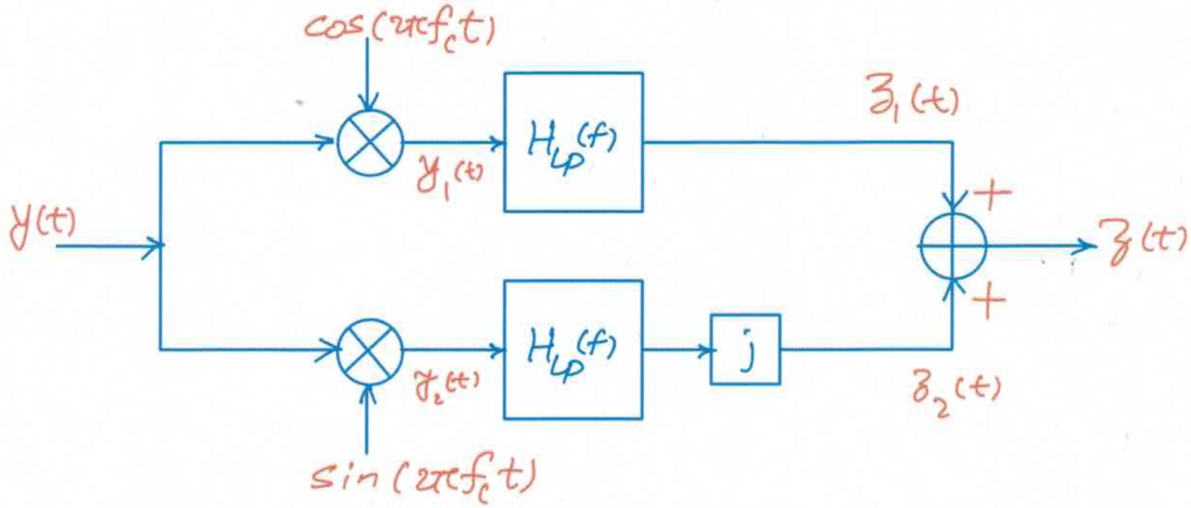
Multiplying by $\cos(2\pi f_c t)$ shifts the spectrum back and forth by f_c. But, as shown, in this scenario, the signal will corrupt itself, and so we cannot recover our original signal $x(t)$.

Note that the LPF will get rid of freq components at $\pm 2f_c$.





d) Now, assume we instead use the following receiver:



where $H_{LP}(f)$ is the same ideal lowpass filter with cutoff at B Hz as shown in part c. Obtain and plot $Z(f) = \mathcal{F}\{z(t)\}$ and show that $z(t)$ will indeed be equal to $x(t)$.

Noting that the LPF's will reject the freq. components at $\pm 2f_c$, and using $Y(f)$ from part b), we can write:

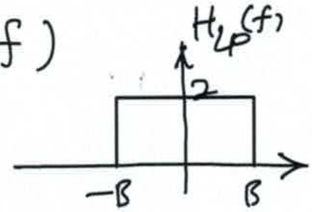
$$Y(f) = \frac{1}{2} X(f - f_c) + \frac{1}{2} X^*(-(f + f_c))$$

$$\Rightarrow Y_1(f) = \frac{1}{2} Y(f - f_c) + \frac{1}{4} Y(f + f_c)$$

$$= \frac{1}{4} X(f - 2f_c) + \frac{1}{4} X(f)$$

$$+ \frac{1}{4} X^*(-f) + \frac{1}{4} X^*(-(f + 2f_c))$$

$$\Rightarrow Z_1(f) = \frac{1}{2} X(f) + \frac{1}{2} X^*(-f)$$



Similarly:

$$Y_2(f) = \frac{1}{2j} Y(f - f_c) - \frac{1}{2j} Y(f + f_c)$$

$$= \frac{1}{4j} X(f - 2f_c) + \frac{1}{4j} X(f)$$

$$- \frac{1}{4j} X^*(-f) - \frac{1}{4j} X^*(-(f + 2f_c))$$

$$\Rightarrow Z_2(f) = \left[\frac{1}{2\cancel{j}} X(f) - \frac{1}{2\cancel{j}} X^*(-f) \right] \cdot \cancel{j}$$

$$\Rightarrow Z_2(f) = \frac{1}{2} X(f) - \frac{1}{2} X^*(-f)$$

$$\Rightarrow Z(f) = Z_1(f) + Z_2(f) = \underline{X(f)}$$