UCLA Electrical Engineering Department

EE102: Signals and Systems

Midterm

Monday, July 20, 2015

Duration: 2hrs.
Total points: 30

Rules:
1. Following items are prohibited during test – calculator, computer, iPad, iPod, cell phones, tablets, programmable watches, closed books, etc.
2. Please mark the answer sheet with your full name and make sure to staple it.

1. **(9 points) Let's start with an appetizer… Signals for now**

(i) (3 pts.) Let's start with Euler as I promised. Simplify following in rectangular form:

a. $x = 2^j$

b. $x = j^j$

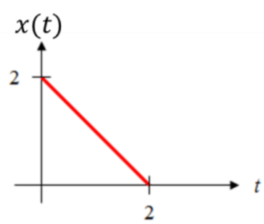
(ii) (3 pts.) Consider the signal $x(t) = 3\sin(4\pi t) + 7\cos(10t)$.

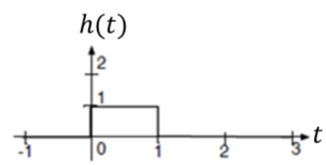
a. Is it periodic or aperiodic? If periodic, determine the fundamental period.

b. Is it continuous or discrete?

c. Is it finite energy or finite power?

(iii) (3 pts.) Graphically convolve $x(t) * h(t)$.





Solution 1:

(i) **Part a –**

$x = 2^j$ or $x = e^{\ln 2^j} = e^{j \ln 2}$

Using Euler, $x = \cos(\ln 2) + j\sin(\ln 2)$

Part b –

Note, $j = e^{j\pi/2}$

Therefore, $x = j^j = e^{\left(\frac{j\pi}{2}\right)j} = e^{-\pi/2}$

(ii) $x(t) = 3\sin(4\pi t) + 7\cos(10t)$

a. Aperiodic, since $\frac{T_1}{T_2} = \frac{1/2}{\pi/5}$ is not a rational number

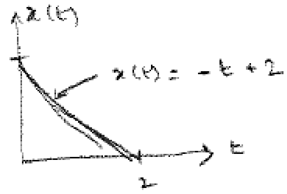
b. Continuous

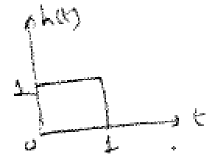
c. Since the signal $x(t)$ is combination of two periodic signals, it has infinite energy. That is, finite power signal.

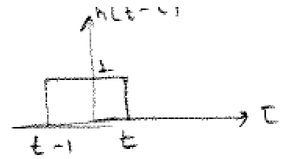
(iii)

1) Part(b)

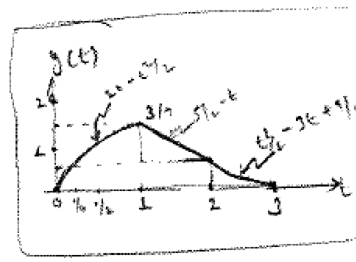
$y(t) = x(t) * h(t)$



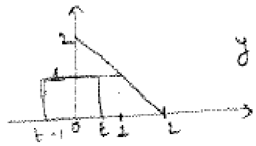




$t < 0, \quad y(t) = 0$



$0 < t < 1$



$$y(t) = \int_0^t (2-\tau)\,d\tau = 2t - \frac{t^2}{2}$$

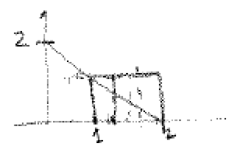
$t=0, \; y=0$

$t=1, \; y=3/2$

$t=1/2, \; y = 1 - \frac{1}{8} = 7/8$

$t=1/4, \; \frac{1}{2} - \frac{1}{32} = \frac{15}{32}$

$1 < t < 2$



$$y(t) = \int_{t-1}^t (2-\tau)\,d\tau = \left[2\tau - \frac{\tau^2}{2}\right]_{t-1}^t = \left[2(t) - \frac{t^2}{2}\right] - \left[2t - 2 - \frac{(t-1)^2}{2}\right]$$

$$= 2t - \frac{t^2}{2} - 2t + 2 + \frac{t^2}{2} + \frac{1}{2} - t = \frac{5}{2} - t$$

$t=1, \; y = 3/2$; $t = 2, \; y = \ldots$

$2 < t < 3$

$$y(t) = \int_{t-1}^2 (2-\tau)\,d\tau = \left[2\tau - \frac{\tau^2}{2}\right]_{t-1}^2$$

$$= 4 - 2 - 2(t-1) + \frac{(t-1)^2}{2}$$

$$= \frac{4 - 4(t-1) + (t-1)^2}{2}$$

$$= 2 - 2(t-1) + \frac{(t-1)^2}{2}$$

$$= 2 - 2t + 2 + \frac{t^2}{2} + \frac{1}{2} - t = \frac{t^2}{2} - 3t + \frac{9}{2}$$

$t = 2: \; y = 2 - 2 + \frac{1}{2} = \frac{1}{2}$

$t = 3: \; y = 2 - 4 + 2 = 0$

$2 - 2 \cdot \frac{3}{2} + \left(\frac{3}{2}\right)^2 \cdot \frac{1}{2}$

$-1 + \frac{9}{8} = 1/8$

2. (12 points) Time for the main course… Welcome systems!

(i) (2 pts.) Consider a system S with $x(\cdot)$ as input, $y(\cdot)$ as output and input-output relationship is $y(t) = x(\alpha t), \;\; t \in (-\infty, \infty)$. Is this system TI or TV? Why?

(ii) (4 pts.) A linear system S s described by the input-output relation:

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)} u(t-\tau) x(\tau)\,d\tau + \int_{-\infty}^{t} x(\tau)\,d\tau, \qquad t \in (-\infty, \infty)$$

Find the impulse response function $h(t,\tau)$ of S. Then compute output $y(t)$ given that the input $x(t) = tu(t)$.

(iii) (6 pts.) A system S_1 is described by

$$b(t) = \int_{-\infty}^{\infty} u(t-\tau) a(\tau) u(\tau)\,d\tau, \;\; t \geq 0$$

where $a(t)$ and $b(t)$ are input and output, respectively. Second system S_2 is described by:

$$\frac{df(t)}{dt} + f(t) = e(t), \qquad t > 0$$

and $f(0) = 0$, where $e(t)$ and $f(t)$ are input and output, respectively. Find output $y(t)$ of the cascaded system $S = S_1 S_2$ given that:

$$x(t) = tu(t) \to [S_1] \to z(t) \to [S_2] \to y(t)$$

Solution:

(i) $y(t) = S[x(t)] = x(\alpha t)$

$$y(t) = x(\alpha t) = \int_{-\infty}^{\infty} \delta(\alpha t - \tau)x(\tau)d\tau = \int_{-\infty}^{\infty} \delta\left(\alpha\left[t - \frac{\tau}{\alpha}\right]\right)x(\tau)d\tau$$

For $\alpha \neq 1$, $h(t,\tau) = \delta\left(\alpha\left[t - \frac{\tau}{\alpha}\right]\right) \neq h(t-\tau) \Longrightarrow S$ is TV

(ii) $y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)}u(t-\tau)x(\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau,\ \ t \in (-\infty, \infty)$

The above can be written as:

$$y(t) = \int_{-\infty}^{\infty} [e^{-(t-\tau)} + 1]u(t-\tau)x(\tau)d\tau$$

Comparing with superposition integral,

$$h(t,\tau) = \left[e^{-(t-\tau)} + 1\right]u(t-\tau) = h(t-\tau)$$

For part-2 of the problem:

$$y(t) = \int_{-\infty}^{\infty} [e^{-(t-\tau)} + 1]u(t-\tau)\tau u(\tau)d\tau$$

$$= \int_0^t \left[e^{-(t-\tau)} + 1\right]\tau d\tau = \left[-1 + t + \frac{t^2}{2} + e^{-t}\right]u(t)$$

(iii) Using convolution integral, the impulse response function of S_1 is given by:

$$h_1(t-\tau) = u(t-\tau)$$

Next, we solve differential equation using integrating factor:

$$f(t) = \int_0^t e^{-(t-\tau)}e(\tau)d\tau, \qquad t > 0$$

Equivalently,

$$f(t) = \int_{-\infty}^{\infty} [e^{-(t-\tau)}u(t-\tau)]e(\tau)u(\tau)d\tau, \qquad t > 0$$

Therefore,

$$h_2(t-\tau) = e^{-(t-\tau)}u(t-\tau)$$

Now,

$$x(t) = tu(t) \to [S_1] \to z(t) \to [S_2] \to y(t)$$

Then,

$$z(t) = \int_{-\infty}^{\infty} u(t-\tau)\tau u(\tau)d\tau = \int_0^t \tau d\tau = \frac{t^2}{2}u(t)$$

Above cascade system can now be written as:

$$\frac{t^2}{2}u(t) \to [S_2] \to y(t)$$

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)}u(t-\tau)\frac{\tau^2}{2}u(\tau)d\tau$$

Therefore,

$$y(t) = \int_0^t e^{-(t-\tau)}\frac{\tau^2}{2}d\tau = \frac{1}{2}(2 - 2t + t^2 - 2e^{-t})$$

Note:
You can compute above by different and simpler method. First, compute the impulse response function $h_{12}(t)$ of the cascaded combination, then use convolution integral to find the output. OR, use Laplace Transform and even go simpler.

3. (9 points) And… Laplace Transform as the dessert

(i) (2 pts.) Find Laplace Transform of $f(t) = \cos(\omega t + \varphi)u(t)$

(ii) (4 pts.) Find $h(t)$ given that its Laplace Transform $H(s)$ is

$$H(s) = \frac{s^2}{s^2 + s + 4}$$

If $H(s)$ is the Transfer Function of a LTIC system, what is the differential equation relating its input $x(t)$ and output $y(t)$.

(iii) (3 pts.) Find $y(t)$ when $x(t) = tu(t)$.

Solution:

(i) $f(t) = \cos(\omega t + \varphi) = \cos\omega t \cos\varphi - \sin\omega t \sin\varphi$

$\mathcal{L}[f(t)] = \mathcal{L}[\cos\omega t]\cos\varphi - \mathcal{L}[\sin\omega t]\sin\varphi$

$= \frac{s}{s^2+\omega^2}\cos\varphi - \frac{\omega}{s^2+\omega^2}\sin\varphi$

(ii) $H(s) = 1 - \frac{s+4}{s^2+s+4} = H_1(s) - H_2(s)$

$H_2(s) = \frac{s+4}{s^2+s+4} = \frac{a + b(s+1/2)}{(s+1/2)^2 + \left(\frac{\sqrt{15}}{2}\right)^2} \Rightarrow (a+b/2) + bs = s+4$ $\boxed{a = 7/2,\ b = 1}$

$\alpha = 1/2,\ \omega = \frac{\sqrt{15}}{2}$

$f_2(t) = \left[\frac{7}{\sqrt{15}}\sin\left(\frac{\sqrt{15}}{2}t\right) + \cos\left(\frac{\sqrt{15}}{2}t\right)\right]e^{-t/2}u(t)$

$$\boxed{h(t) = \delta(t) - \left[\frac{7}{\sqrt{15}}\sin\left(\frac{\sqrt{15}}{2}t\right) + \cos\left(\frac{\sqrt{15}}{2}t\right)\right]e^{-t/2}u(t)}$$

$(s^2+s+4)Y(s) = (s^2)X(s)$

$$\boxed{\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) + 4y(t) = \frac{d^2}{dt^2}x(t)}$$

(iii) $X(s) = 1/s^2$ $\quad Y(s) = \frac{1}{s^2+s+4} = \frac{a + b(s+1/2)}{(s+1/2)^2 + (\sqrt{15}/2)^2}$ $\quad a = 1,\ b = 0$

$\alpha = 1/2,\ \omega = \sqrt{15}/2$

$$\boxed{y(t) = \frac{2}{\sqrt{15}}e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right)u(t)}$$