UCLA Electrical Engineering Department

EE102: Signals and Systems

FINAL EXAMINATION

Summer 2015

Wednesday, August 12, 2015

Exam Duration: 3 hrs.
Total: 50 points

NAME:

	1	2	3	4	5	Total
Marks Obtained						
Maximum Marks	**12**	**10**	**9**	**9**	**10**	**50**

Instructions

1. Closed Book, Calculators, Cell Phones and IPods Are NOT Allowed
2. A Two-Sided Sheet (8.5 x 11.0) of Notes IS Allowed
3. If you use extra sheet, then make sure to staple them with these examination papers

PLEASE MAKE SURE THAT YOU HAVE 11 PAGES OF
THESE EXAMINATION PAPERS

NUMBER OF YOUR SUBMITTED PAGES: [ ]

1. **(12 pts) Xmas Gift – Wrapped & Unwrapped**

(i) (2.5 pts) A system with input signal $x(t)$ and output signal $y(t)$ can be: **Linear**, **Time-invariant** and/or **Causal**. Determine which of these properties hold and which do not hold for continuous-time system $y(t) = [A + x(t)]\cos(\omega_c t)$. Justify your answer.

(ii) (4.5 pts) Determine if each of the following statements about LTI systems is *true* or *false*. Write a sentence justifying your answer. Just answering True or False will only get you half a point.

(a) If an LTI system is causal, then it is stable.

(b) The cascade of a noncausal LTI system with a causal LTI system must be noncausal.

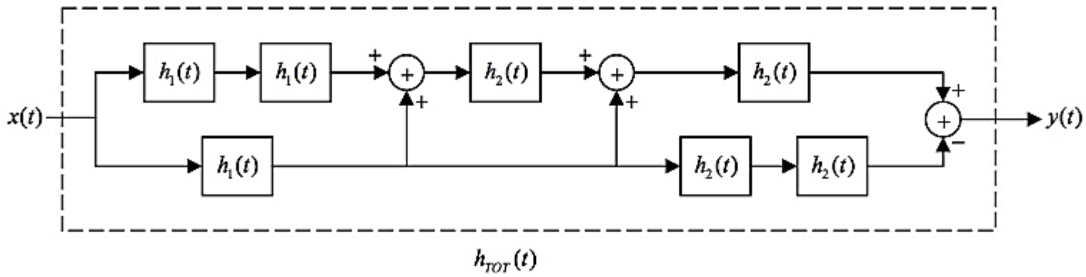
(c) Time-reversing the input to a LTI system results in a time-reversed output.

(iii) (2 pts) What is the Nyquist rate for the signal $f(t) = sinc(200t) + sinc^2(200t)$?

(iv) (3 pts) Evaluate the following integral with the help of Fourier transform theorems.

$$\int_{-\infty}^{\infty} \left(sinc(t)\right)^4 dt$$

2. **(10 pts) Revisiting LTI systems – Laplace to Fourier**
Note: (i) and (ii) are standalone questions
(i) (6 pts) Consider the following interconnections of LTI systems $h_1(t)$ and $h_2(t)$.



(a) (2 pts) Express the overall system response $h_{TOT}(t)$ in terms of the impulse responses $h_1(t)$ and $h_2(t)$.

(b) (3 pts) Assume that the system 1 in inverse of system 2, i.e. $h_1(t) * h_2(t) = \delta(t)$. Simplify your answer to (a).

(c) (1 pt) From your answer in (b), what is the purpose of overall system?

(ii) (4 pts) The PZP (Poles-Zeroes-Plot) of the Laplace Transform $H(s)$, of the IRF $h(t)$, of a LTIC system S, has the following characteristics:

zero of order 1 at : 3
pole of order 1 at : -1
pole of order 2 at : -2

$$H(0) = -\frac{3}{2}$$

Find $h(t)$.

3. **(9 pts) Fourier Series and MSE**
   (i) (3 pts) Find the Fourier series expansion, the amplitude and phase spectra of the periodic signal $f(t) = \cos(\pi t)$.

   (ii) (2 pts) Compute the mean square error when $f(t)$ is approximated by
   $$F_0 + F_1 e^{j\pi t} + F_{-1} e^{-j\pi t}$$

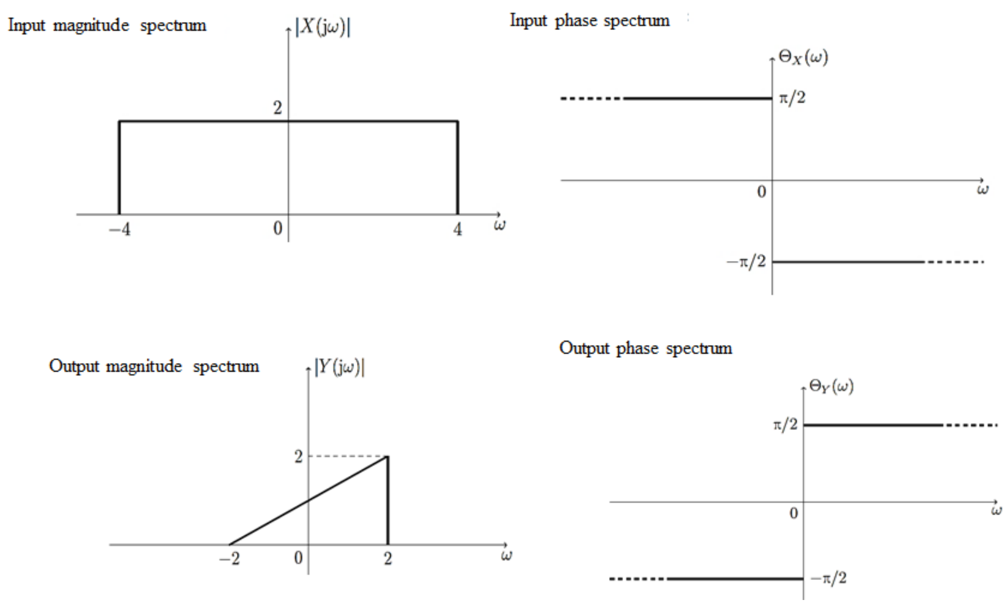
(iii) (4 pts) The signal above is applied to the linear system whose system function is

$$H(s) = \frac{2s}{2s+1}$$

Find the amplitude and phase spectra of the output.

4. **(9 pts) Fourier Transform and Parseval**
   (i) (1.5 pts) What is/are the condition(s) for a function to have both Laplace and Fourier Transforms? Explain with an example.

   (ii) (3.5 pts) Let $h(t)$, $x(t)$, and $y(t)$, for $-\infty < t < \infty$, be the impulse response function, the input, and the output of a linear time-invariant system, respectively. Give the following spectra:



Find $H(\omega)$ from the above spectra and from the fact that $H(\omega) = 0$ for ω not belonging to the interval $(-2,2)$. Find the impulse response function $h(t)$ from $H(\omega)$ found above. Is the system causal?

(iii) (4 pts) Find the energy in the signal $f(t) = e^{-at}u(t)$, and find the bandwidth W such that 95% of the energy is contained in frequencies below W for $a = 1$. (Leave the solution in fraction form if required)

5. **(10 pts) Half-wave symmetry**

Let $f(t)$ be a periodic signal of period 1. One says that $f(t)$ has half-wave symmetry if

$$f\left(t-\frac{1}{2}\right)=-f(t)$$

(a) (4 pts) Does $f(t)=\sin(2\pi t)+\sin(4\pi t)$ have half-wave symmetry? Find its Fourier series.

(b) (6 pts) If $f(t)$ has period 1, has half-wave symmetry and its Fourier series is

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{j2n\pi t}$$

show that $c_n=0$ if n is even.

Solution of Final Exam

Summer 2015

P1. (i) $y(t) = [A + x(t)] \cos(\omega_c t)$

- Linear: if $A \neq 0$, then it is <u>not</u> linear.
  if $A = 0$, then it is linear.

- Time-invariant:

$$x_1(t) = x(t-t_0) \rightarrow y_1(t) = [A + x_1(t)] \cos(\omega_c t)$$
$$= [A + x(t-t_0)] \cos(\omega_c t)$$
$$\neq [A + x(t-t_0)] \cos(\omega_c (t-t_0))$$

Therefore, $y_1(t) \neq y(t-t_0)$. So it is not time-invariant

- Causal: $y(t)$ depends on $x(t)$, so it is causal.

(ii) (a) False. Example: $h(t) = u(t)$.

$h(t)$ is causal, since $h(t) = 0$, for $t < 0$.

But $\int |h(t)| dt \rightarrow \infty$. and it is not stable.

(b) False

$$x(t) \rightarrow \boxed{h_1(t)} \rightarrow \boxed{h_2(t)} \rightarrow y(t)$$

$h(t) = h_1(t) * h_2(t)$

h_1 causal, h_2 non-causal.

Consider $h_1(t) = u(t-2)$ and $h_2(t) = \delta(t+1)$

$\Rightarrow h(t) = h_1(t) * h_2(t) = u(t-2) * \delta(t+1) = u(t-1)$

$\Rightarrow h(t)$ is causal.

(c) False $\quad y(t) = x(t) * h(t)$

$$g(t) = x(-t) * h(t)$$

$$\Rightarrow Y(\omega) = X(\omega)\, H(\omega)$$

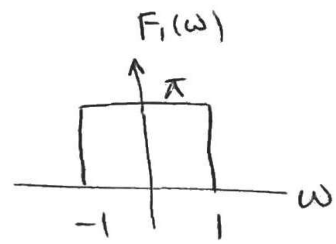
$$G(\omega) = X(-\omega)\, H(\omega) \neq Y(-\omega)$$

$$\Rightarrow g(t) \neq y(-t)$$

(iii) Consider $f_1(t) = \text{Sinc}(t)$ and $f(t) = (f_1(t))^4 = (\text{Sinc}(t))^4$

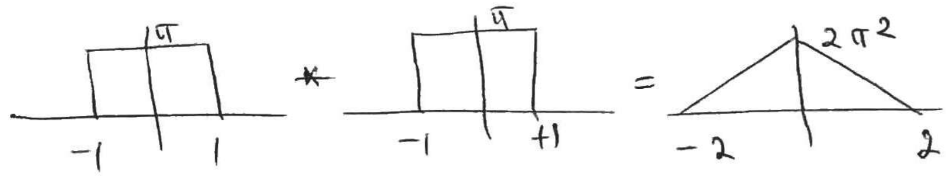
and $\text{Sinc}(t) = \dfrac{\sin(t)}{t}$.

$$F(\omega) = \left(\frac{1}{2\pi}\right)^2 F_1(\omega) * F_1(\omega) * F_1(\omega) * F_1(\omega)$$

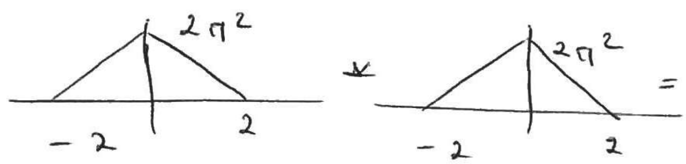


$$F_1(\omega) = \mathcal{F}\{f_1(t)\} = \mathcal{F}\{\text{Sinc}(t)\} = \pi\, \text{rect}\left(\frac{\omega}{2}\right)$$

$F_1(\omega) * F_1(\omega) \longrightarrow$

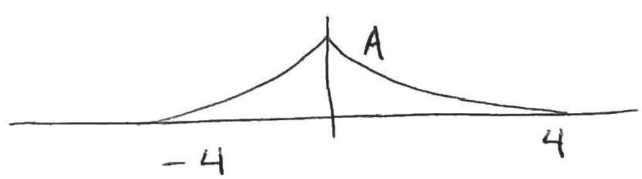


$(F_1(\omega) * F_1(\omega)) * (F_1(\omega) * F_1(\omega)) \rightarrow$



$$A = 2\int_0^2 (2\pi^2 - \pi^2 t)^2\, dt$$

$$= \frac{16\pi^4}{3}$$



$$\Rightarrow \int_{-\infty}^{\infty} (\text{sinc}(t))^4\, dt = \int_{-\infty}^{\infty} f(t)\, dt = F(\omega)\Big|_{\omega=0} = F(0) = \frac{1}{4\pi^4} A = \frac{4}{3}$$

Problem 1

part (iii)

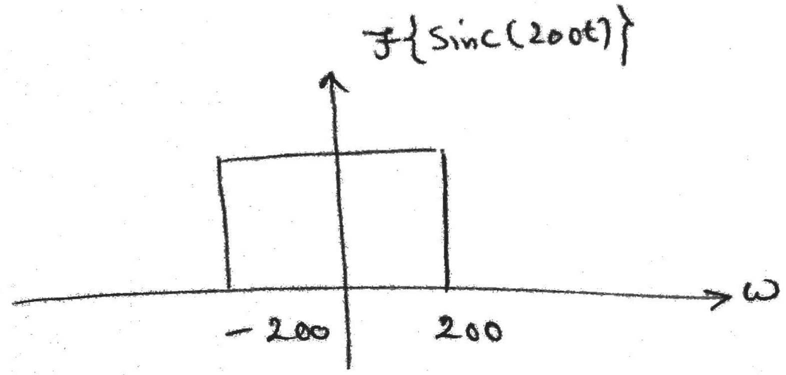
$$f(t) = \text{sinc}(200t) + \text{sinc}^2(200f).$$

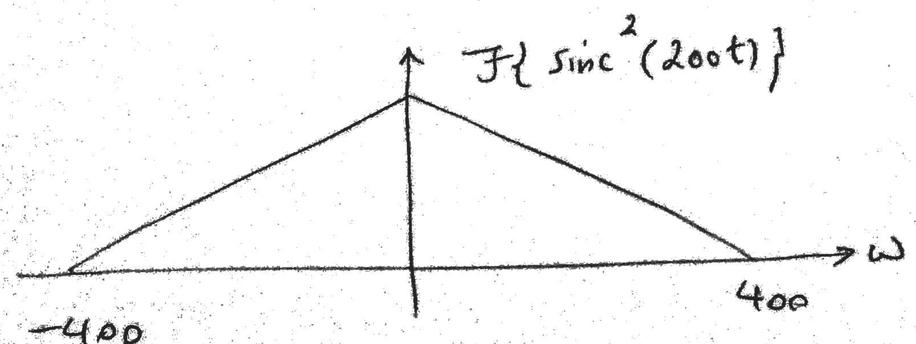
Bandwidth of sinc (200t) is 200.

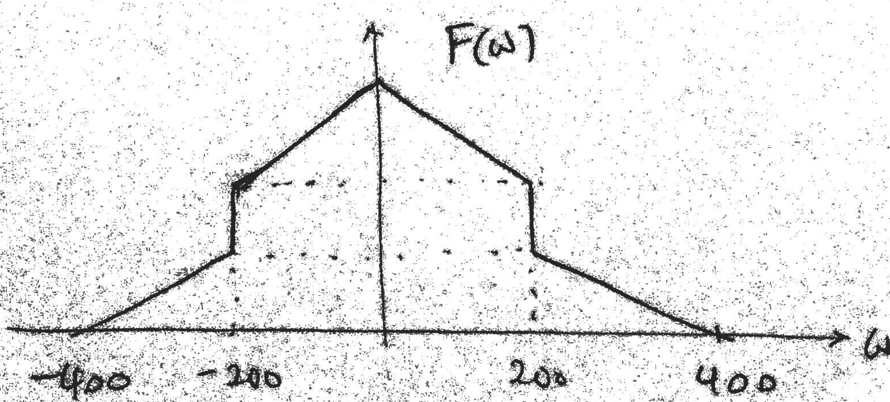
Bandwidth of $\text{sinc}^2(200t)$ is 400.

Therefore, bandwidth of $f(t)$ is 400.

So, Nyquist rate is $2 \times 400 = 800$ Hz

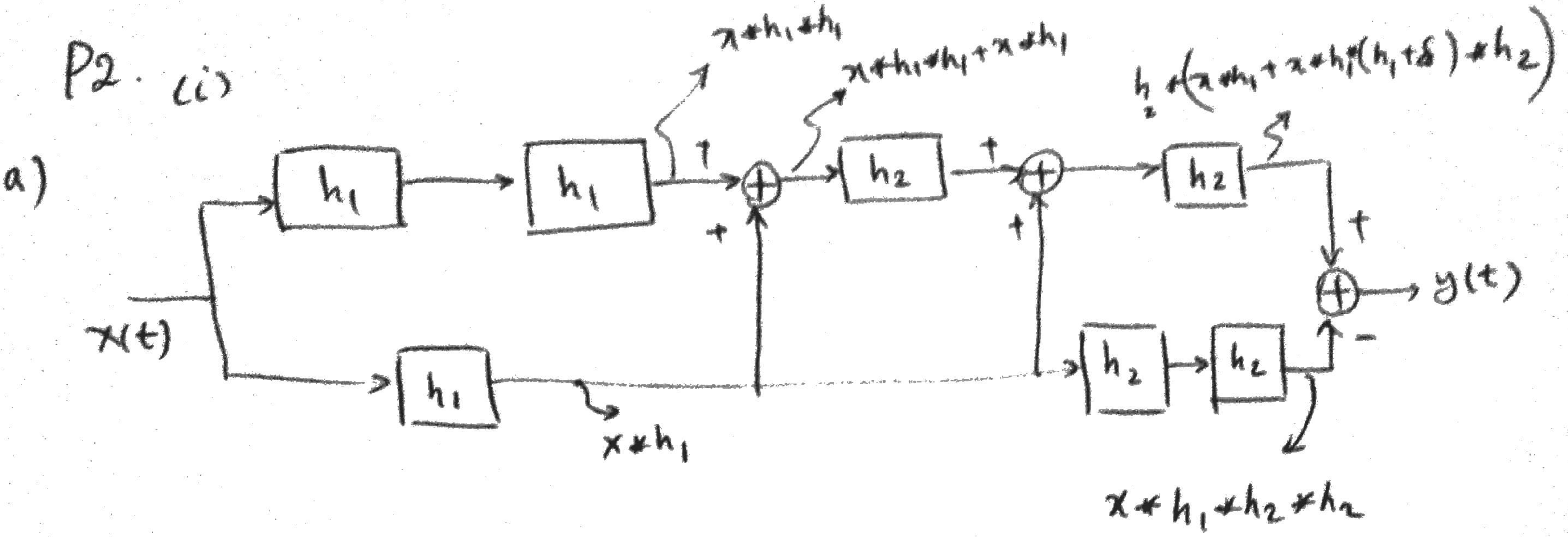






P2. (i)

a)



$y(t) = x*h_1*h_2 + x*h_1*h_1*h_2*h_2 + x*h_1*h_2*h_2$
$\quad - x*h_1*h_2*h_2$
$= x*h_1*h_2 + x*h_1*h_1*h_2*h_2$

$\Longrightarrow h_{TOT}(t) = h_1(t)*h_2(t) + h_1(t)*h_2(t)*h_1(t)*h_2(t)$

(b) if $h_1(t)*h_2(t) = \delta(t)$, then
$h_{TOT}(t) = \delta(t) + \delta(t)*\delta(t) = 2\delta(t)$

(c) The system doubles the input. $y(t) = 2x(t)$.

(ii) $H(s) = \dfrac{2(s-3)}{(s+1)(s+2)^2}$ $\quad$ ROC $= \{Re(s) > -1\}$ since LTIC

$$H(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} = \frac{(A+B)s^2 + (4A+3B+C)s + (4A+2B+C)}{(s+1)(s+2)^2}$$

$$\Rightarrow \begin{cases} A+B=0 \\ 4A+3B+C=2 \\ 4A+2B+C=-6 \end{cases} \Rightarrow A=-8,\ B=8,\ C=10$$

$$\Rightarrow h(t) = \mathcal{L}^{-1}(H(s)) = -8e^{-t}u(t) + 8e^{-2t}u(t) + 10te^{-2t}u(t).$$

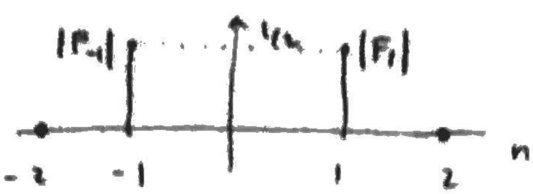
P3. (i) $f(t) = \cos(\pi t)$

$$T = 2, \quad \omega_0 = \frac{2\pi}{2} = \pi$$

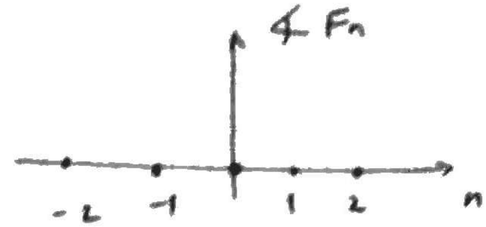
$$f(t) = \cos(\pi t) = \frac{1}{2} e^{j\pi t} + \frac{1}{2} e^{-j\pi t} = \sum_{n=-\infty}^{\infty} F_n e^{jn\pi t}$$

$\Longrightarrow$ $F_1 = \frac{1}{2}$, $F_{-1} = \frac{1}{2}$, $F_n = 0$, for $n \neq \pm 1$

$\Longrightarrow$



amplitude plot



phase plot

(ii) $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$ $\qquad \bar{f}(t) = F_0 + F_1 e^{j\pi t} + F_{-1} e^{-j\pi t}$

(So $\omega_0 = \pi$) $\qquad mse = \frac{1}{T} \int_T \left(f(t) - \bar{f}(t) \right)^2 dt$

$$= \frac{1}{T} \int_T \left\{ \sum_{\substack{n=-\infty \\ n \neq 0, \pm 1}}^{\infty} F_n e^{jn\pi t} \right\}^2 dt$$

parseval Theorem $= \displaystyle\sum_{\substack{n=-\infty \\ n \neq 0, \pm 1}}^{\infty} |F_n|^2 = \sum_{n=-\infty}^{\infty} |F_n|^2 - \left\{ |F_0|^2 + |F_1|^2 + |F_{-1}|^2 \right\}$

$$= \frac{1}{T} \int_T (f(t))^2 dt - \left(|F_0|^2 + |F_1|^2 + |F_{-1}|^2 \right)$$

(iii) $H(s) = \dfrac{2s}{2s+1}$,

Consider ROC : $\{ s : Re(s) > -\frac{1}{2} \}$. Therefore, ROC includes $s = j\omega$.

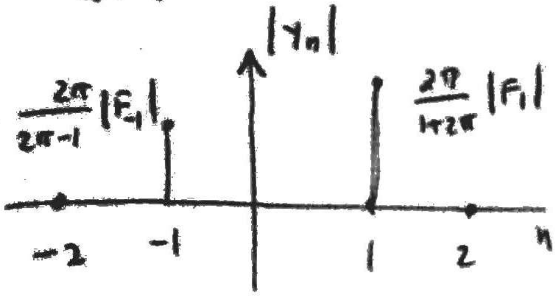
$$H(\omega) = \frac{2\omega}{2\omega+1}$$

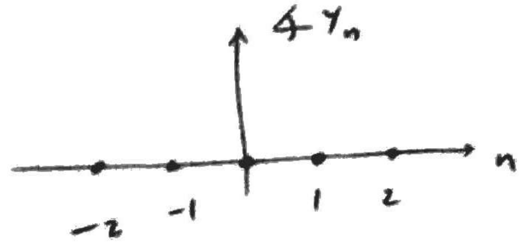
$$e^{j\omega t} \rightarrow \boxed{H(\omega)} \rightarrow H(\omega) e^{j\omega t}$$

$$\Rightarrow y(t) = F_0 H(0) + F_1 H(\pi) e^{j\pi t} + F_{-1} H(-\pi) e^{-j\pi t}$$

$$= \frac{2\pi}{2\pi+1} F_1 e^{j\pi t} - \frac{2\pi}{1-2\pi} F_{-1} e^{-j\pi t}.$$

$$Y_1 = \frac{2\pi}{2\pi+1} F_1 \quad , \quad Y_2 = \frac{-2\pi}{1-2\pi} F_{-1}$$





P4. (i) A signal $x(t)$ has a Laplace Transform whenever it is of exponential order. That is, there must be a real number B such that

$$\lim_{t \to \infty} \left| x(t) e^{-Bt} \right| = 0.$$

For example, every exponential signal $x(t) = Ae^{\alpha t}$ has Laplace Transform.

A signal $x(t)$ has Fourier transform whenever it is absolutely integrable. That is,

$$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty.$$

For example, $x(t) = e^{-t} u(t)$ has Fourier transform $X(\omega) = \frac{1}{1+j\omega}$, since $\int_{-\infty}^{\infty} |x(t)| \, dt = \int_{0}^{\infty} e^{-t} dt = 1 < \infty.$

(iii) Consider two complex numbers $\alpha = r_1 e^{j\theta_1}$ and $\beta = r_2 e^{j\theta_2}$.

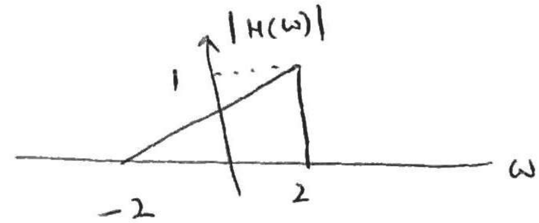
where r_1 and r_2 are amplitudes and θ_1, θ_2 are phases.

$\alpha \cdot \beta = r_1 r_2 e^{j(\theta_1+\theta_2)}$. Therefore, $\alpha\beta$ has amplitude $r_1 r_2$

and phase $\theta_1 + \theta_2$.

$Y(\omega) = H(\omega) X(\omega)$, and $H(\omega) = 0$ for $\omega \in (-2, 2)$

$\Longrightarrow$



$\measuredangle H(\omega) = \measuredangle Y(\omega) - \measuredangle X(\omega) \Rightarrow \measuredangle H(\omega) = 0$, for all ω and

$H(\omega)$ is pure real and negative. $\Rightarrow H(\omega) = \begin{cases} -\left(\frac{1}{4}\omega + \frac{1}{2}\right) & -2 \le \omega \le 2 \\ 0 & \text{oth.} \end{cases}$

$$h(t) = \mathcal{F}^{-1}\{H(\omega)\} = \frac{1}{2\pi} \int_{-2}^{2} -\left(\frac{1}{4}\omega + \frac{1}{2}\right) e^{j\omega t}\, d\omega$$

$$= \frac{-1}{2\pi} \left\{ \frac{1}{4}\left(\frac{1}{jt}\omega e^{j\omega t} + \frac{1}{t^2} e^{j\omega t} \right)\Bigg|_{-2}^{2} + \frac{1}{2jt} e^{j\omega t} \Bigg|_{-2}^{2} \right\}$$

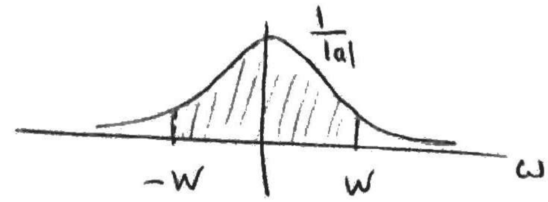
$$= \frac{-1}{8\pi}\left(\frac{2}{j} e^{j2t} - \frac{2}{j} e^{-j2t} + \frac{1}{t^2} e^{j2t} - \frac{1}{t^2} e^{-j2t} \right) - \frac{1}{4\pi j t}\left(e^{j2t} - e^{-j2t}\right)$$

$$= \frac{-1}{8\pi}\left(4\sin(2t) + \frac{2j}{t^2}\sin(2t) \right) - \frac{1}{2\pi t}\sin(2t)$$

$\Longrightarrow$ not causal.

(iii) $\mathcal{E}_f = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_0^{\infty} e^{-2at} dt = \frac{-1}{2a} e^{-2at} \Big|_0^{\infty} = \frac{1}{2a}, \quad a>0$

$F(\omega) = \frac{1}{a+j\omega} \Rightarrow |F(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$:



$\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = 2\pi \int |f(t)|^2 dt = \frac{2\pi}{2a} = \frac{\pi}{a}$

Set $a=1$. For what W, $\int_{-W}^{W} |F(\omega)|^2 d\omega = 0.95\pi$?

$\int_{-W}^{W} \left(\frac{1}{\sqrt{\omega^2+1}} \right)^2 d\omega = \int_{-W}^{W} \frac{1}{1+\omega^2} d\omega = \tan^{-1}(\omega) \Big|_{-W}^{W} = 2\tan^{-1}(W)$

$2\tan^{-1}(W) = 0.95\pi \Rightarrow W = \tan\left(0.95 * \frac{\pi}{2}\right) \simeq 12.7$ Hz

P5.

(a) $f(t) = \sin(2\pi t) + \sin(4\pi t)$, $T=1$, $\omega_0 = 2\pi$

$$f(t-\tfrac{1}{2}) = \sin\left(2\pi(t-\tfrac{1}{2})\right) + \sin\left(4\pi(t-\tfrac{1}{2})\right)$$
$$= \sin(2\pi t - \pi) + \sin(4\pi t - 2\pi)$$
$$= -\sin(2t) + \sin(4t)$$
$$\neq -f(t)$$

$f(t)$ is <u>not</u> halfway symetric.

$$f(t) = \frac{1}{2j} e^{j2\pi t} - \frac{1}{2j} e^{-j2\pi t} + \frac{1}{2j} e^{j4\pi t} - \frac{1}{2j} e^{-j4\pi t}$$

$F_1 = F_2 = \frac{1}{2j}$, $F_{-1} = F_{-2} = -\frac{1}{2j}$, $F_n = 0$, $n \neq \pm 1, \pm 2$

(b) $$f(t-\tfrac{1}{2}) = \sum_n C_n e^{j2n\pi(t-\frac{1}{2})} = \sum_n C_n e^{-jn\pi} e^{j2\pi n t}$$

$$= \sum_n C_n (-1)^n e^{j2n\pi t}$$, $$f(t-\tfrac{1}{2}) = -f(t) = \sum_n -C_n e^{j2\pi n t}$$

$\Rightarrow$ $C_n (-1)^n = -C_n$, so if n is even, $-C_n = C_n \Rightarrow C_n = 0$.