EE 102 Midterm → Paganini
Winter 05

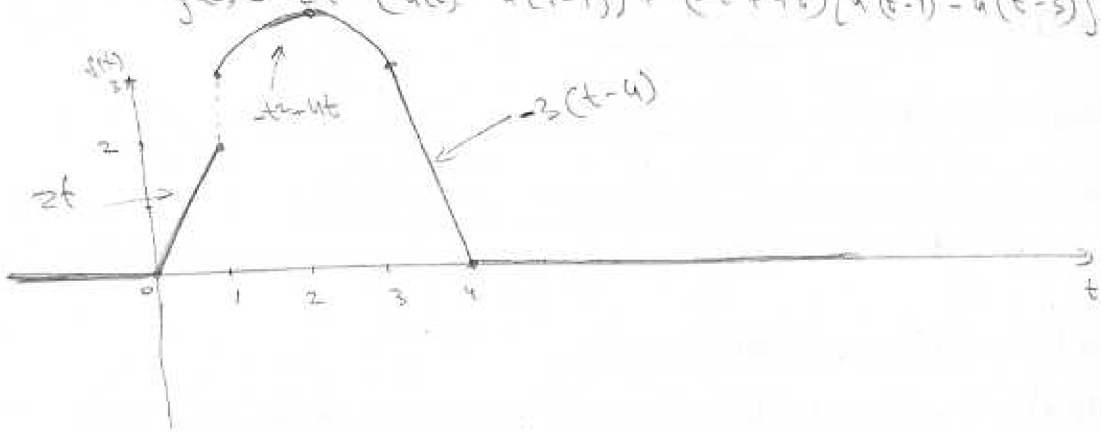
Problem 1 [15 pts]

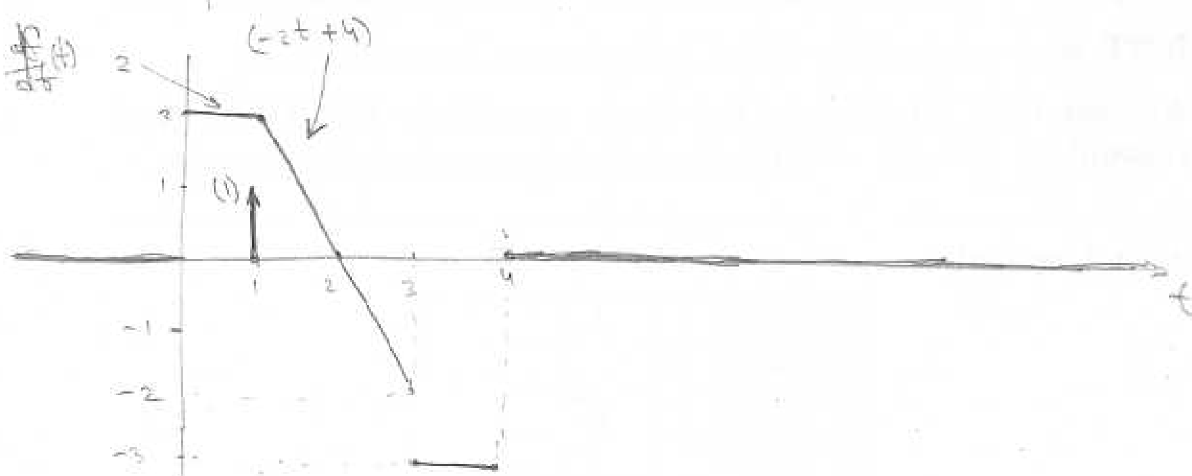
For the function

$$f(t) = 2tu(t)u(1-t) + (-t^2+4t)u(t-1)u(3-t) - 3(t-4)u(t-3)u(4-t).$$

Sketch the functions $f(t)$, and $\frac{df}{dt}$, and give an analytic formula for the latter in its simplest form.

$f(t) = 2t\,(u(t) - u(t-1)) + (-t^2+4t)[u(t-1) - u(t-3)] - 3(t-4)[u(t-3) - u(t-4)]$





$$\boxed{\frac{df}{dt}(t) = 2[u(t) - u(t-1)] + \delta(t-1) + (-2t+4)[u(t-1) - u(t-3)] - 3[u(t-3) - u(t-4)]}$$

$$h(t,\tau) = T[\delta(t-\tau)] = \int_{-\infty}^{\infty} e^{(t-\sigma)}\delta(\sigma-\tau)\,d\sigma = e^{(t-\tau)}\int_{-\infty}^{\infty}\delta(\sigma-\tau)\,d\sigma = \boxed{e^{(t-\tau)}}$$

Problem 2 [20 pts]

Consider a linear system with input-output relationship

$$y(t) = \int_{-\infty}^{\infty} e^{(t-\sigma)}x(\sigma)d\sigma$$

a) Find the impulse response function $h(t,\tau)$.

b) Is the system time invariant? Causal?

c) Now take the input to be $x(t) = tu(t)$. Find the output $y(t)$.

or forward way

$$y(t) = \int_{-\infty}^{\infty} h(t,\sigma)x(\sigma)d\sigma = \int_{-\infty}^{\infty} e^{(t-\sigma)}x(\sigma)\,d\sigma$$

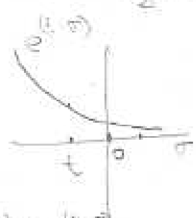
a) $\Rightarrow$ by comparison $\boxed{h(t,\tau) = e^{(t-\tau)}}$

b) $\rightarrow$ depends only on $\underline{t-\tau}$ $\leftarrow$ $\boxed{\text{Time invarient}}$

$\rightarrow$ when $t<\tau$ $h(t,\tau)\neq 0$ $\leftarrow$ $\boxed{\text{Non-causal}}$

$x(t) = t\,u(t)$

$$y(t) = \int_{-\infty}^{\infty} e^{(t-\sigma)}\sigma\, u(\sigma)\,d\sigma = \int_0^{\infty} e^{(t-\sigma)}\sigma\,d\sigma$$



$$y(t) = \begin{cases} t\leq 0 & \int_0^{\infty} e^{(t-\sigma)}\sigma\,d\sigma \\ t>0 & \int_0^{\infty} e^{(t-\sigma)}\sigma\,d\sigma \end{cases}$$

$\leftarrow$ Looks all the same for all t

$$y(t) = \int_0^{\infty} e^{(t-\sigma)}\sigma\,d\sigma = -\sigma e^{(t-\sigma)} - e^{(t-\sigma)}\Big|_0^{\infty} =$$

$$\int e^{(-\sigma+t)}\sigma\,d\sigma = -\sigma e^{(-\sigma+t)} + \int e^{-\sigma+t}\,d\sigma =$$

$v = -e^{-\sigma+t}$ $du = d\sigma$

$$= -\sigma e^{(-\sigma+t)} - e^{(-\sigma+t)}$$

$$= (\cancel{0} - \cancel{0}) - (-e^{t}) = \boxed{e^{t}}$$

c) $\boxed{y(t) = e^{t}}$

Problem 3 [15 pts]

Consider the differential equation

$$\frac{dy(t)}{dt} + y(t) = t, \qquad y(0) = 1.$$

Find the solution $y(t)$ for $t \geq 0$. use laplace

$s Y(s) - \cancel{y(0^-)}^{1} + Y(s) = \frac{1}{s^2}$

$(s+1) Y(s) = \frac{s^2+1}{s^2}$

$Y(s) = \frac{s^2+1}{(s+1)s^2} = \frac{A}{s+1} + \frac{B}{s} + \frac{C}{s^2}$

$A = (s+1) Y(-1) = 2 = A$ $\qquad$ $C = s^2 Y(0) = 1 = C$

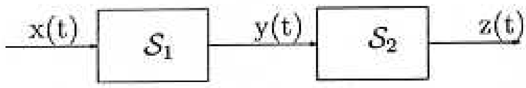
$\lim_{s \to \infty} sY(s) = A + B = \lim_{s \to \infty} \frac{s^2+1}{s^2+s} = 1 = A + B = 2 + B$

$B = -1$

$Y(s) = \frac{2}{s+1} - \frac{1}{s} + \frac{1}{s^2}$

$\boxed{y(t) = (2e^{-t} + t - 1)u(t)}$ for $t \geq 0$

Problem 4 [20 pts]



Consider the interconnection of linear, time-invariant and causal systems $\mathcal{S}_1$ and $\mathcal{S}_2$ shown above. Let $g_1(t)$ and $g_2(t)$ denote respectively the **step** responses of $\mathcal{S}_1$, $\mathcal{S}_2$.

15 (a) For the special case $g_1(t) = u(t)[1 - e^{-t}]$, $g_2(t) = u(t)e^{-2t}$, find the step response $g_{12}(t)$ of the cascade.

2 (b) Now for the general case, give a formula for $g_{12}(t)$ as a function of $g_1(t)$ and $g_2(t)$.

$$g(t) = T[u(t)] = \int_{-\infty}^{t} h(\sigma)d\sigma \quad \Rightarrow \quad \frac{dg}{dt} = h$$

$$h_1 = \frac{dg_1}{dt} = \delta(t)\cancel{(1 - e^{-t})} + u(t)e^{-t} = u(t)e^{-t} \qquad e^{0} = 1 \quad \to 0$$

$$h_2 = \frac{dg_2}{dt} = \delta(t)e^{-2t} - 2u(t)e^{-2t} = \delta(t) - 2e^{-2t}u(t)$$

$h_{1,2} = h_2 * h_1$ → LTI causal → use Laplace

$$H_1 = \frac{1}{s+1} \qquad H_2 = 1 - \frac{2}{s+2} = \frac{s+2-2}{s+2} = \frac{s}{s+2}$$

$$H_{12} = H_1 H_2 = \frac{s}{(s+1)(s+2)}$$

$g_{12} = \int_{-\infty}^{t} h_{12}(\sigma)d\sigma$, ⇒ use laplace property 4 $\quad G_{12} = \frac{H_{12}(s)}{s} = \frac{1}{(s+1)(s+2)}$

$$G_{12} = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \qquad (s+1)G(-1) = A = 1 \qquad B = (s+2)G(-2) = -1$$

$$= \frac{1}{s+1} - \frac{1}{s+2}$$

$$\Rightarrow \boxed{g_{12}(t) = \left(e^{-t} - e^{-2t}\right)u(t)} \qquad 8$$

a)

Problem 5 [30 pts]

Consider the system described by the input-output relationship:

$$y(t) = \begin{cases} x(t) & \text{if } x(t) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

8 a) Is the system (i) linear? (ii) time invariant? (iii) causal?

9 b) We apply the input $x(t) = -t^2 + 2t$; sketch $y(t)$, $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$.

6 c) Find the Laplace transform $Y(s)$ for the output $y(t)$ in part b), and give its domain of convergence.

a) Suppose $x(t_0) = 3$ $\quad y(t_0) = 3 = T[x(t_0)]$

$T[(-1)x(t_0)] = T[(-1)*3] = \emptyset \neq (-1)\,y(t_0)$

by definition $(-1)\,x(t_0) < 0$

$\Rightarrow$ $\boxed{\text{Not Linear}}$

$$y(t-\tau) = \begin{cases} x(t-\tau) & \text{if } x(t-\tau) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$T[x(t-\tau)] = \begin{cases} x(t-\tau) & \text{if } x(t-\tau) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow y(t-\tau) = T[x(t-\tau)]$

$\Rightarrow$ $\boxed{\text{Time Invariant}}$

$\boxed{\text{Causal}}$, $y(t)$ depends on present $x(t)$