EE102 - Winter 2004 - Midterm Solutions

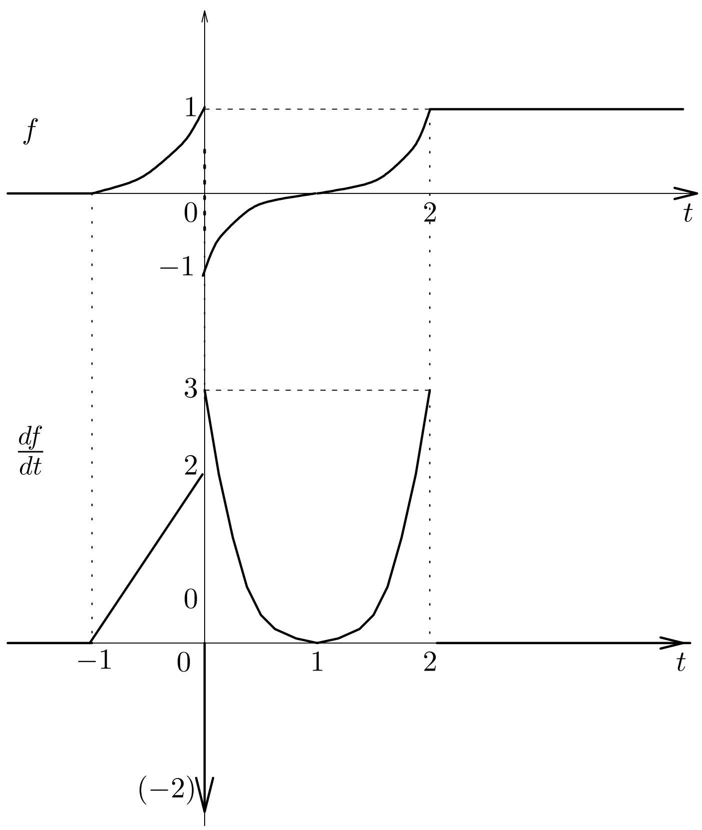
Problem 1 [15 pts]

For the function

$$f(t) = (t+1)^2[u(t+1) - u(t)] + (t-1)^3 u(t)u(2-t) + u(t-2).$$

Sketch $f(t)$ and $\dfrac{df}{dt}$, and give an analytic formula for the latter in its simplest form.

Solution:



$$\frac{df}{dt} = 2(t+1)[u(t+1) - u(t)] + 3(t-1)^2[u(t) - u(t-2)] - 2\delta(t).$$

Problem 2 [20 pts]

We are given a linear system defined by the input-output relationship

$$y = x + f * x,$$

where $*$ denotes convolution and the function $f(t) = e^{-|t|}$.

(a) Is the system time invariant? Causal? Find its impulse response function.

(b) Find the response to the input $x(t) = u(t) - u(t-2)$.

Solution:

(a) The system is **time-invariant**. We can check this by definition, using

$$y(t) = x(t) + \int_{-\infty}^{\infty} f(t-\sigma)x(\sigma)d\sigma, \tag{1}$$

$$\begin{aligned} \implies y(t-\tau) &= x(t-\tau) + \int_{-\infty}^{\infty} f(t-\tau-\sigma)x(\sigma)d\sigma \\ &= x(t-\tau) + \int_{-\infty}^{\infty} f(t-v)x(v-\tau)dv = T[x(t-\tau)]. \end{aligned}$$

Another way would be to note that the system is really defined by a convolution:

$$y = x + f * x = \delta * x + f * x = (\delta + f) * x = h * x,$$

where

$$h(t) = \delta(t) + f(t) = \delta(t) + e^{-|t|}.$$

Therefore the system must be time-invariant, with the above $h(t)$ its impulse response.

Since $h(t)$ is not zero for $t < 0$, the system is **non-causal**. We could also deduce this by definition, since all values of $x(t)$ (past, present and future) are involved in the integral of (1).

(b) We must compute the convolution

$$\begin{aligned}(f * x)(t) &= \int_{-\infty}^{\infty} x(t-\sigma) f(\sigma) d\sigma \\ &= \int_{-\infty}^{\infty} [u(t-\sigma) - u(t-\sigma-2)] f(\sigma) d\sigma \\ &= \int_{t-2}^{t} e^{-|\sigma|} d\sigma.\end{aligned}$$

Since

$$e^{-|\sigma|} = \begin{cases} e^{\sigma} & \text{for } \sigma < 0 \\ e^{-\sigma} & \text{for } \sigma \geq 0 \end{cases},$$

we solve the integral by breaking it in cases according to the value of t:

$$\int_{t-2}^{t} e^{-|\sigma|} d\sigma = \begin{cases} \int_{t-2}^{t} e^{\sigma} d\sigma & \text{for } t < 0 \\ \\ \int_{t-2}^{0} e^{\sigma} d\sigma + \int_{0}^{t} e^{-\sigma} d\sigma & \text{for } 0 \leq t < 2 \\ \\ \int_{t-2}^{t} e^{-\sigma} d\sigma & \text{for } 2 \leq t \end{cases}$$

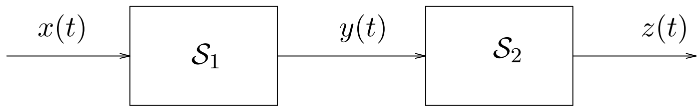
$$= \begin{cases} e^{t} - e^{t-2} & \text{for } t < 0 \\ \\ 1 - e^{t-2} + 1 - e^{-t} & \text{for } 0 \leq t < 2 \\ \\ e^{-(t-2)} - e^{-t} & \text{for } 2 \leq t \end{cases}$$

$$\begin{aligned} &= u(-t)[e^{t} - e^{t-2}] + [u(t) - u(t-2)] \cdot [2 - e^{t-2} - e^{-t}] + u(t-2)[e^{-(t-2)} - e^{-t}] \\ &= 2[u(t) - u(t-2)] + u(-t)e^{t} - e^{t-2}[u(-t) + u(t) - u(t-2)] \\ &\quad + e^{-t}[-u(t) + u(t-2) - u(t-2)] + u(t-2)e^{-(t-2)} \\ &= 2[u(t) - u(t-2)] + u(-t)e^{t} - u(2-t)e^{t-2} - u(t)e^{-t} + u(t-2)e^{-(t-2)}. \end{aligned}$$

Now, adding the extra term $x(t)$, we get (either expression is a valid answer)

$$\begin{aligned} y(t) &= u(-t)[e^{t} - e^{t-2}] + [u(t) - u(t-2)] \cdot [3 - e^{t-2} - e^{-t}] + u(t-2)[e^{-(t-2)} - e^{-t}] \\ &= 3[u(t) - u(t-2)] + u(-t)e^{t} - u(2-t)e^{t-2} - u(t)e^{-t} + u(t-2)e^{-(t-2)}. \end{aligned}$$

Problem 3 [15 pts]



Consider the cascade interconnection of the figure, where $\mathcal{S}_1$ is an integrator, and $\mathcal{S}_2$ is an LTI system with impulse response

$$h_2(t) = \delta(t) + 3e^{2t}u(t).$$

We apply a certain input $x(t)$, and obtain the output $z(t) = u(t)(-1 + e^{2t})$.
Determine $x(t)$ and $y(t)$.

Solution:

We use Laplace transforms since both systems are causal, and the given output $z(t)$ is zero for $t < 0$. Its transform is

$$Z(s) = -\frac{1}{s} + \frac{1}{s-2} = \frac{-s+2+s}{s(s-2)} = \frac{2}{s(s-2)}.$$

The transfer function of $\mathcal{S}_2$ is

$$H_2(s) = 1 + \frac{3}{s-2} = \frac{s+1}{s-2},$$

so we get

$$Y(s) = \frac{Z(s)}{H_2(s)} = \frac{2}{s(s+1)} = 2\left(\frac{1}{s} - \frac{1}{s+1}\right).$$

Therefore

$$y(t) = 2u(t)(1 - e^{-t}).$$

Since $\mathcal{S}_1$ is an integrator, then

$$x(t) = \frac{dy}{dt} = 2u(t)e^{-t}.$$

This could also be done by Laplace, noting that $H_1(s) = \frac{1}{s}$, and computing

$$X(s) = \frac{Y(s)}{H_1(s)} = \frac{2}{s+1}.$$

Problem 4 [20 pts]

Solve the differential equation

$$\frac{d^2 f(t)}{dt^2} - 2\frac{df(t)}{dt} + 2f(t) = 2, \qquad t \geq 0,$$

with initial conditions

$$f(0-) = 0, \quad \frac{df(t)}{dt}(0-) = 1.$$

Solution:

Applying Laplace, and its derivative property to the differential equation, we have

$$\left[s^2 F(s) - f(0-)s - \frac{df}{dt}(0-)\right] - 2\left[sF(s) - f(0-)\right] + 2F(s) = \mathcal{L}[2] = \frac{2}{s}.$$

Using the given initial conditions leads to

$$(s^2 - 2s + 2)F(s) = 1 + \frac{2}{s} = \frac{s+2}{s}.$$

Therefore

$$F(s) = \frac{s+2}{s(s^2-2s+2)} = \frac{A}{s} + \frac{Ms+N}{s^2-2s+2}.$$

Multiply by s, limit as $s \to 0$ gives

$$\left.\frac{s+2}{s^2-2s+2}\right|_{s=0} = \boxed{1 = A.}$$

Multiply by s, limit as $s \to \infty$ gives

$$\left.\frac{s+2}{s^2-2s+2}\right|_{s\to\infty} = 0 = A + M, \implies \boxed{M = -1.}$$

One more equation, e.g. set $s = 1$, gives

$$\left.\frac{s+2}{s(s^2-2s+2)}\right|_{s=1} = 3 = A + M + N, \implies \boxed{N = 3.}$$

Therefore

$$F(s) = \frac{1}{s} + \frac{-s+3}{s^2-2s+2} = \frac{1}{s} + \frac{-(s-1)+2}{(s-1)^2+1},$$

that leads, using Laplace table and properties, to the solution

$$f(t) = u(t)\left[1 + e^t(-\cos(t) + 2\sin(t))\right].$$

Problem 5 [30 pts]

Consider the system described by the input-output relationship

$$y(t) = \int_{t-1}^{t} \sigma x(\sigma) d\sigma. \tag{2}$$

(a) Is the system (i) linear? (ii) time invariant? (iii) causal?

(b) We apply the input $x(t) = u(t)\sin(t)$; find $y(t)$.

(c) Find the Laplace transform $Y(s)$ for the output $y(t)$ in part (b), and its DOC.

Solution:

(a) The system is linear:

$$\begin{aligned} T[\alpha_1 x_1(t) + \alpha_2 x_2(t)] &= \int_{t-1}^{t} \sigma[\alpha_1 x_1(\sigma) + \alpha_2 x_2(\sigma)] d\sigma. \\ &= \alpha_1 \int_{t-1}^{t} \sigma x_1(\sigma) d\sigma + \alpha_2 \int_{t-1}^{t} \sigma x_2(\sigma) d\sigma. \\ &= \alpha_1 T[x_1(t)] + \alpha_2 T[x_2(t)]. \end{aligned}$$

It is time-varying:

$$\begin{aligned} y(t-\tau) &= \int_{t-\tau-1}^{t-\tau} \sigma x(\sigma) d\sigma \\ &= \int_{t-1}^{t} (v-\tau) x(v-\tau) dv \\ &\neq \int_{t-1}^{t} v x(v-\tau) dv = T[x(t-\tau)]. \end{aligned}$$

Since the integral in the definition (2) for $y(t)$ only involves values of $x(\sigma)$ in the interval $[t-1, t]$ (past and present), the system is causal.

Another way to solve this would be to rewrite (2) as

$$y(t) = \int_{-\infty}^{\infty} \sigma[u(\sigma - t + 1) - u(\sigma - t)] x(\sigma) d\sigma,$$

which has the form of a superposition intergral, and therefore the system impulse response function must be

$$h(t, \sigma) = \sigma[u(\sigma - t + 1) - u(\sigma - t)].$$

Since it depends on both t and σ (not just the difference), and it is zero for $t < \sigma$, this implies the system is time-varying and causal.

(b) To compute

$$y(t) = \int_{t-1}^{t} \sigma u(\sigma) \sin(\sigma) d\sigma,$$

one way is to discriminate the cases

$$\int_{t-1}^{t} \sigma u(\sigma) \sin(\sigma) d\sigma = \begin{cases} 0 & \text{for } t < 0, \\ \int_0^t \sigma \sin(\sigma) d\sigma & \text{for } 0 \leq t < 1, \\ \int_{t-1}^t \sigma \sin(\sigma) d\sigma & \text{for } 1 \leq t. \end{cases}$$

Using integration by parts gives

$$\int \sigma \sin(\sigma) d\sigma = -\sigma \cos(\sigma) + \int \cos(\sigma) = -\sigma \cos(\sigma) + \sin(\sigma),$$

therefore the above integrals give

$$\begin{aligned} y(t) &= \begin{cases} 0 & \text{for } t < 0, \\ [-\sigma \cos(\sigma) + \sin(\sigma)]_0^t & \text{for } 0 \leq t < 1, \\ [-\sigma \cos(\sigma) + \sin(\sigma)]_{t-1}^t & \text{for } 1 \leq t. \end{cases} \\ &= \begin{cases} 0 & \text{for } t < 0, \\ -t \cos(t) + \sin(t) & \text{for } 0 \leq t < 1, \\ -t \cos(t) + \sin(t) + (t-1) \cos(t-1) - \sin(t-1) & \text{for } 1 \leq t. \end{cases} \end{aligned}$$

Writing it in one equation gives

$$\begin{aligned} y(t) &= [u(t) - u(t-1)] \cdot [-t \cos(t) + \sin(t)] \\ &\quad + u(t-1)[-t \cos(t) + \sin(t) + (t-1) \cos(t-1) - \sin(t-1)] \\ &= u(t)[-t \cos(t) + \sin(t)] - u(t-1)[-(t-1) \cos(t-1) + \sin(t-1)]. \end{aligned} \tag{3}$$

Another way would be to first note (e.g. from observing the cases) the identity:

$$y(t) = u(t) \int_0^t \sigma \sin(\sigma) d\sigma - u(t-1) \int_0^{(t-1)} \sigma \sin(\sigma) d\sigma.$$

Here the function

$$f(t) = u(t) \int_0^t \sigma \sin(\sigma) d\sigma = u(t) \left[-t \cos(t) + \sin(t) \right]$$

(using the same integration by parts), so we have

$$y(t) = f(t) - f(t-1), \tag{4}$$

that gives the same answer as (3).

Yet another way, that was actually suggested by a couple of people in the class, is to do the integration by parts directly in the original integral. Integrating by parts twice gives:

$$
\begin{aligned}
\int \sigma u(\sigma)\sin(\sigma)d\sigma &= -\sigma u(\sigma)\cos(\sigma) + \int \cos(\sigma)u(\sigma)d\sigma \\
&= -\sigma u(\sigma)\cos(\sigma) + \sin(\sigma)u(\sigma) - \int \sin(\sigma)\delta(\sigma)d\sigma \\
&= u(\sigma)\left[-\sigma\cos(\sigma) + \sin(\sigma)\right].
\end{aligned}
$$

For the last step, note that $\sin(\sigma)\delta(\sigma) = \sin(0)\delta(\sigma) = 0$.

Now incrementing the previous function between the limits $t-1$ and t gives the same answer as (3).

(c) The best way is to first transform the function $f(t) = u(t)[-t\cos(t) + \sin(t)]$. Using the Laplace properties we find that

$$
\begin{aligned}
F(s) &= -\left[-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right] + \frac{1}{s^2+1} \\
&= \frac{s^2+1-s(2s)}{(s^2+1)^2} + \frac{1}{s^2+1} \\
&= \frac{1-s^2+s^2+1}{(s^2+1)^2} \\
&= \frac{2}{(s^2+1)^2}
\end{aligned}
$$

Now using the delay property (note $f(t)$ already contains a step $u(t)$),

$$
\mathcal{L}[f(t-1)] = \mathcal{L}[u(t-1)f(t-1)] = e^{-s}F(s).
$$

Therefore from (4) we have

$$
Y(s) = F(s) - e^{-s}F(s) = \frac{2(1-e^{-s})}{(s^2+1)^2}.
$$

The transform has poles at $s = \pm i$. Therefore the domain of convergence is $Re[s] > 0$.