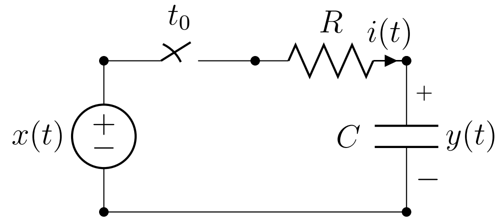
1. Consider the system constructed by the following R-C circuit;



where $R = 1\,\Omega$ and $C = 0.2\,\mathrm{F}$. Assume $t_0 = 0$ and $y(0) = 0$.

(a) (5 points) A capacitor has a voltage that is proportional to the charge, which is equal to the integral of the current. And the capacitance C, is defined as the ratio of charge $q(t)$ on each conductor to the voltage $v(t)$ between them: $C = q(t)/v(t)$. Considering these relationships, write a differential equation describing the given system, where $y(t) = v(t)$.

(b) (10 points) What is the impulse response, $h(t;\tau)$, of this system.

(c) (10 points) Is this system linear, time-invariant, causal? Explain your answers.

(d) (5 points) What is the output of the system when the input is $x(t) = 5t\mathrm{u}(t)$? Compute the output in the **time domain**.

Solution:

(a) The voltage-current relationship in a capacitor is as following:

$$v(t) = \frac{1}{C}q(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,\mathrm{d}\tau,$$
$$\Rightarrow i(t) = C\frac{\mathrm{d}}{\mathrm{d}t}v(t).$$

Applying Kirchhoff's voltage law,

$$Ri(t) + y(t) = x(t), \quad i(t) = C\frac{\mathrm{d}}{\mathrm{d}t}y(t).$$

Therefore, the resulting differential equation is,

$$RC\frac{\mathrm{d}}{\mathrm{d}t}y(t) + y(t) = x(t), \quad t > t_0$$

Using the given values,

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 5y(t) = 5x(t), \quad t > 0$$

(b) By multiplying both sides by e^{5t}, the differential equation can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\{e^{5t}y(t)\} = 5e^{5t}x(t)$$

Integrating both sides from 0 to t results in

$$e^{5t}y(t) - e^{5\cdot 0}y(0) = \int_0^t 5e^{5\sigma}x(\sigma)\,\mathrm{d}\sigma$$

or

$$y(t) = 5e^{-5t}\int_0^t e^{5\sigma}x(\sigma)\,\mathrm{d}\sigma$$

The impulse response $h(t;\tau)$ is the output of the system to the input $\delta(t-\tau)$. Therefore,

$$\begin{aligned}
h(t;\tau) &= 5e^{-5t}\int_0^t e^{5\sigma}\delta(\sigma-\tau)\,\mathrm{d}\sigma \\
&= 5e^{-5t}\int_{-\infty}^{\infty} e^{5\sigma}u(\sigma)u(t-\sigma)\delta(\sigma-\tau)\,\mathrm{d}\sigma \\
&= 5e^{-5(t-\tau)}u(t-\tau)
\end{aligned}$$

(c) The system is linear since the output is the integration of a linear function of the input. Or, for any k_1, k_2 and x_1, x_2,

$$\begin{aligned}
T[k_1x_1(t)+k_2x_2(t)] &= 5e^{-5t}\int_0^t e^{5\sigma}(k_1x_1(\sigma)+k_2x_2(\sigma))\,\mathrm{d}\sigma \\
&= k_1\cdot 5e^{-5t}\int_0^t e^{5\sigma}x_1(\sigma)\,\mathrm{d}\sigma + k_2\cdot 5e^{-5t}\int_0^t e^{5\sigma}x_2(\sigma)\,\mathrm{d}\sigma \\
&= k_1T[x_1(t)] + k_2[x_2(t)].
\end{aligned}$$

It's time-invariant since $h(t;\tau)$ is only a function of $t-\tau$. It's also verified that $T[x(t-\tau)] = y(t-\tau)$ for any τ.

Moreover, the system is causal as $y(t)$ depends on $x(\sigma)$ for $\sigma \le t$.

(d)

$$\begin{aligned}
y(t) &= 5e^{-5t}\int_0^t e^{5\sigma}[5\sigma u(\sigma)]\,\mathrm{d}\sigma \\
&= 5^2e^{-5t}\left(\left[\frac{1}{5}\sigma e^{5\sigma}\right]_{\sigma=0}^{t} - \int_0^t \frac{1}{5}e^{5\sigma}\,\mathrm{d}\sigma\right) \\
&= [5e^{-5t}te^{5t} - (e^{5t}-1)]u(t) \\
&= [-e^{5t}+5t+1]u(t)
\end{aligned}$$

Please go to next page...

2. (15 points) The following are inputs to a time-invariant system $\mathcal{S}_2$ and the corresponding outputs:

$$x_1(t) = \mathrm{u}(t) - \mathrm{u}(t-2) \quad \rightarrow \quad y_1(t) = \mathrm{e}^{-t}\mathrm{u}(t) - \mathrm{e}^{-2t}\mathrm{u}(t-1),$$
$$x_2(t) = \mathrm{u}(t) - \mathrm{u}(t-1) \quad \rightarrow \quad y_2(t) = -\mathrm{e}^{-t}\mathrm{u}(t).$$

Can the system said to be linear? Why?

Solution:

$$x_1(t) - x_2(t-1) = \mathrm{u}(t) - \mathrm{u}(t-1) = x_1(t).$$

Assume that the system is linear. As it's also time-invariant, from the prior formula, we should have

$$T[x_1(t) - x_2(t-1)] = T[x_2(t)],$$

i.e.,

$$y_1(t) - y_2(t-1) = y_2(t).$$

However, the left-hand side is

$$\text{LHS} = \mathrm{e}^{-t}\mathrm{u}(t) + (\mathrm{e}^{-(t-1)} - \mathrm{e}^{-2t})\mathrm{u}(t-1),$$

and the right-hand side is

$$\text{RHS} = -\mathrm{e}^{-t}\mathrm{u}(t),$$

which are not equal to each other, a contradiction. So the system is not linear.