Solutions

Multiple-choice questions – Check all the answers that apply.

1. (a) (3 points) Consider the system described by the input-output relation

$$y(t) = x(t+2) - x(t-2).$$

What are its properties?

✓ **Linear**
✓ **Time-invariant**
☐ Causal

Solution: Linear: $x(t) \to k_1 x_1(t) + k_2 x_2(t)$, then output$= k_1(x_1(t+2) - x_1(t-2)) + k_2(x_2(t+2) - x_2(t-2))$. Time-invariant: $x(t) \to x(t-\tau)$, then output$= x(t+2-\tau) - x(t-2-\tau) = y(t-\tau)$. Non-causal: output at time t depends on the *future*, $x(t+2)$.

(b) (3 points) Consider the system described by the input-output relation

$$y(t) = x(t)x(t-1).$$

What are its properties?

☐ Linear
✓ **Time-invariant**
✓ **Causal**

Solution: Nonlinear: $x(t) \to k_1 x_1(t) + k_2 x_2(t)$, then $y \neq k_1 y_1(t) + k_2 y_2(t)$. Time-invariant: $x(t) \to x(t-\tau)$, then output$= x(t-\tau)x(t-1-\tau) = y(t-\tau)$. Causal: output at time t depends only on input at present and past values of t.

(c) (3 points) Consider the system described by the input-output relation

$$y(t) = \int_{-\infty}^{t+1} x(\sigma)\, \mathrm{d}\sigma.$$

What are its properties?

✓ **Linear**
✓ **Time-invariant**
☐ Causal

Solution: Linear: $x(t) \to k_1 x_1(t) + k_2 x_2(t)$, then $y = k_1 y_1(t) + k_2 y_2(t)$. Time-invariant: $x(t) \to x(t-\tau)$, then output$= \int_{-\infty}^{t+1} x(\sigma - \tau)\, \mathrm{d}\sigma = \int_{-\infty}^{t-\tau+1} x(u)\, \mathrm{d}u = y(t-\tau)$ (upon change of variable $\sigma - \tau = u$). Non-causal: output at time t also depends on values of input in the future, at times larger than t.

(d) (1 point) Suppose

$$\mathcal{L}[f](s) = F(s) = \frac{1 - \mathrm{e}^{1}\mathrm{e}^{-s}}{(s^2+1)(s^2-1)}$$

The ROC of $F(s)$ is

- ✓ $\mathrm{Re}(s) > 0$
- ☐ $\mathrm{Re}(s) > 1$
- ☐ $\mathrm{Re}(s) > -1$
- ☐ $0 < \mathrm{Re}(s) < 1$

Solution:

$$\frac{1 - \mathrm{e}^{1}\mathrm{e}^{-s}}{(s^2+1)(s^2-1)} = \frac{1 - \mathrm{e}^{-(s-1)}}{(s^2+1)(s+1)(s-1)}$$

Since

$$\lim_{s\to 1} \frac{1 - \mathrm{e}^{-(s-1)}}{(s-1)} = \lim_{s\to 1} \frac{\mathrm{e}^{-(s-1)}}{1} = 1$$

$s = 1$ is not a pole. The real poles are $s = -1$, $s = \mathrm{j}$, and $s = -\mathrm{j}$. The ROC is therefore the right half plane.

2. A linear system is described by the differential equation

$$\tan(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = x(t), \quad t \geq 0, \quad y(0) = 0$$

Assume all inputs are such $x(t) = 0$, for $t < 0$.

(a) (5 points) Find the system's impulse response function $h(t;\tau)$.

Solution: Multiplying both sides of the equation by $\cos(t)$ we get:

$$\sin(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \cos(t)y(t) = \cos(t)x(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\sin(t)y(t)] = \cos(t)x(t)$$

$$\sin(t)y(t) = \int_0^t \cos(\tau)x(\tau)\ \mathrm{d}\tau.$$

$$y(t) = \int_{-\infty}^{\infty} \frac{\cos(\tau)}{\sin(t)}\mathrm{u}(t-\tau)x(\tau)\ \mathrm{d}\tau, \quad \text{for all } t \text{ where } \sin(t) \neq 0.$$

$$\therefore h(t;\tau) = \frac{\cos(\tau)}{\sin(t)}\mathrm{u}(t-\tau)$$

(b) (5 points) Is the system time-invariant? Is it a causal system?

Solution: $h(t;\tau)$ is not a function of only $(t-\tau)$, which means the system is NOT time-invariant. However, the system is causal since for $t - \tau < 0$, $h(t;\tau)$ is zero.

(c) (5 points) Let $x(t) = \mathrm{u}(t)$. Find the output $y(t)$.

Solution:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} \frac{\cos(\tau)}{\sin(t)}\mathrm{u}(t-\tau)\mathrm{u}(\tau)\ \mathrm{d}\tau \\ &= \frac{1}{\sin(t)}\int_0^t \cos(\tau)\ \mathrm{d}\tau \\ &= \frac{\sin(t)}{\sin(t)} \\ &= 1 \quad t \geq 0 \end{aligned}$$

3. (5 points) When $x_1(t) = e^{-(t-1)}\mathrm{u}(t-1)$ is the input to linear, time-invariant, causal system, $\mathcal{S}$, the corresponding output is $y_1(t) = \mathrm{e}^{-(t-1)}\sin(t-1)\mathrm{u}(t-1)$. When the input is $x_2(t) = \delta(t) - \mathrm{e}^{-t}\mathrm{u}(t)$, the corresponding output is $y_2(t) = \delta(t) + \mathrm{e}^{-t}(\cos(t) - \sin(t))\mathrm{u}(t)$. What is the impulse response, $h_1(t)$ of system $\mathcal{S}$? Do not use the Laplace transform. Use the properties of the system.

Solution: Note that $x_2(t) + x_1(t+1) = \delta(t)$, therefore, since the system is linear and time-invariant, $h_1(t) = y_2(t) + y_1(t+1) = \delta(t) + e^{-t}(\cos(t) - \sin(t))\mathrm{u}(t) + e^{-t}\sin(t)\mathrm{u}(t) = \delta(t) + \mathrm{e}^{-t}\cos(t)\mathrm{u}(t)$.

4. (5 points) A linear, time-invariant system, has step response given by $g(t) = \mathrm{e}^{-t}\mathrm{u}(t) - \mathrm{e}^{-2t}\mathrm{u}(t)$. Determine the output of this system, $y(t)$, given an input $x(t) = \delta(t - \pi) - \cos(\sqrt{3})\mathrm{u}(t)$.

Solution: It is known that $h(t) = \frac{\mathrm{d}}{\mathrm{d}t}g(t) = -\mathrm{e}^{-t}\mathrm{u}(t) + \mathrm{e}^{-t}\delta(t) + 2\mathrm{e}^{-2t}\mathrm{u}(t) - \mathrm{e}^{-2t}\delta(t) = (-\mathrm{e}^{-t} + 2\mathrm{e}^{-2t})\mathrm{u}(t)$. The output $y(t)$ is thus given by

$$\begin{aligned} y(t) &= h(t-\pi) - \cos(\sqrt{3})g(t) \\ &= (-\mathrm{e}^{-(t-\pi)} + 2\mathrm{e}^{-2(t-\pi)})\mathrm{u}(t-\pi) - \cos(\sqrt{3})(e^{-t} - e^{-2t})\mathrm{u}(t). \end{aligned}$$

5. (5 points) Use properties of Laplace transform to derive the scaling property of the delta function.

$$\delta(\alpha t) = ?$$

Assume $\alpha > 0$.

Solution: Using the scaling property of Laplace transform, $\mathcal{L}[f(\alpha t)] = \frac{1}{\alpha}F(\frac{s}{\alpha})$, we know that

$$\mathcal{L}\{\delta(\alpha t)\} = \frac{1}{|\alpha|}$$

Also,

$$\mathcal{L}\left[\frac{1}{\alpha}\delta(t)\right] = \frac{1}{\alpha}\mathcal{L}[\delta(t)] = \frac{1}{\alpha}$$

Since both the Laplace transform of $\delta(\alpha t)$ and $\frac{1}{\alpha}\delta(t)$ equal $\frac{1}{\alpha}$,

$$\delta(\alpha t) = \frac{1}{\alpha}\delta(t).$$