1. *Consider the system described by the following input-output relation:*

$$y(t) = \int_{t-1}^{t} \sigma e^{-(t-\sigma)} x(\sigma) d\sigma, \quad -\infty < t < \infty.$$

(a) *Compute the impulse response $h(t, \tau)$.*
Setting $x(t) = \delta(t - \tau)$, then we have

$$\begin{aligned} h(t, \tau) &= \int_{t-1}^{t} \sigma e^{-(t-\sigma)} \delta(\sigma - \tau) d\sigma \\ &= \int_{-\infty}^{\infty} U(\sigma - t + 1) U(t - \sigma) \sigma e^{-(t-\sigma)} \delta(\sigma - \tau) d\sigma \\ &= U(\tau - t + 1) U(t - \tau) \tau e^{-(t-\tau)}. \end{aligned}$$

(b) *Is the system linear? Is it causal? Is it time-invariant? Explain* ***briefly*** *each of your answers.*
The system is linear (easy to prove by replacing $x(t)$ with a linear combination of signals). It is causal because $h(t, \tau) = 0$ for $t < \tau$. It is time-varying, because $h(t, \tau)$ is not a function of $(t - \tau)$.

(c) *What is the output of the system to $x(t) = \delta(t)$?*

$$T[\delta(t)] = h(t, 0) = 0.$$

(d) *What is the output of the system to $x(t) = U(t)$?*
Since

$$y(t) = \int_{t-1}^{t} \sigma e^{-(t-\sigma)} U(\sigma) d\sigma,$$

then we have

1. When $t < 0$, $y(t) = 0$.
2. When $0 < t < 1$, the integral becomes

$$y(t) = \int_{0}^{t} \sigma e^{-(t-\sigma)} d\sigma = e^{-t} \int_{0}^{t} \sigma e^{\sigma} d\sigma = e^{-t} \left[\sigma e^{\sigma} - e^{\sigma} \right] \Big|_{0}^{t} = t - 1 + e^{-t}.$$

3. When $t > 1$, the integral turns into

$$\begin{aligned} y(t) &= \int_{t-1}^{t} \sigma e^{-(t-\sigma)} d\sigma = e^{-t} \int_{t-1}^{t} \sigma e^{\sigma} d\sigma = e^{-t} \left[\sigma e^{\sigma} - e^{\sigma} \right] \Big|_{t-1}^{t} \\ &= \left(1 - e^{-1}\right) t + 2e^{-1} - 1. \end{aligned}$$

2. *Consider the system $\mathcal{S}_1$ described by the input-output relation:*

$$\mathcal{S}_1: \quad y(t) = \int_{-\infty}^{t} \sigma x(\sigma) d\sigma, \quad -\infty < t < \infty.$$

The impulse response of system $\mathcal{S}_2$, $h_2(t)$ is given by

$$h_2(t) = U(t-1) - U(t-2), \quad -\infty < t < \infty.$$

Also, consider the two following serial configurations:

$$\mathcal{S}_{1,2}: \quad \rightarrow \boxed{\mathcal{S}_1} \rightarrow \boxed{\mathcal{S}_2} \rightarrow$$
$$\mathcal{S}_{2,1}: \quad \rightarrow \boxed{\mathcal{S}_2} \rightarrow \boxed{\mathcal{S}_1} \rightarrow$$

(a) *Compute the impulse response of system $\mathcal{S}_1$, $h_1(t,\tau)$.*
Setting $x(t) = \delta(t-\tau)$, then we have

$$h_1(t,\tau) = \int_{-\infty}^{\infty} U(t-\sigma)\sigma\delta(\sigma-\tau)d\sigma = U(t-\tau)\tau.$$

(b) *Is $\mathcal{S}_1$ linear? Is it causal? Is it time-invariant?* ***Briefly*** *justify your answers.*
The system is linear (prove it by replacing $x(t)$ with a linear combination of signals). It is causal, because $h_1(t,\tau) = 0$ for $t < \tau$. It is time-varying, because $h_1(t,\tau)$ is not a function of $(t-\tau)$.

(c) *Is $\mathcal{S}_2$ linear? Is it causal? Is it time-invariant?* ***Briefly*** *justify your answers.*
The system is linear (defined through an impulse response), causal ($h_2(t) = 0$ for $t < 0$) and time-invariant (the impulse response has only one argument).

(d) *Compute the impulse responses of systems $\mathcal{S}_{1,2}$ and $\mathcal{S}_{2,1}$, i.e., $h_{1,2}(t,\tau)$ and respectively $h_{2,1}(t,\tau)$.*
We'll use $h_2(t,\tau) = h_2(t-\tau)$, and then

$$\begin{aligned}
h_{1,2}(t,\tau) &= \int_{-\infty}^{\infty} h_2(t,\sigma)h_1(\sigma,\tau)d\sigma \\
&= \int_{-\infty}^{\infty} [U(t-\sigma-1) - U(t-\sigma-2)]\tau U(\sigma-\tau)d\sigma \\
&= \int_{-\infty}^{\infty} \tau U(t-\sigma-1)U(\sigma-\tau)d\sigma - \int_{-\infty}^{\infty} \tau U(t-\sigma-2)U(\sigma-\tau)d\sigma \\
&= \left(\tau\int_{\tau}^{t-1} d\sigma\right) U(t-1-\tau) - \left(\tau\int_{\tau}^{t-2} d\sigma\right) U(t-2-\tau) \\
&= \tau(t-1-\tau)U(t-1-\tau) - \tau(t-2-\tau)U(t-2-\tau) \\
&= \begin{cases} 0, & \text{when } t-\tau < 1, \\ \tau(t-1-\tau), & \text{when } 1 < t-\tau < 2, \\ \tau, & \text{when } t-\tau > 2. \end{cases}
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
h_{2,1}(t,\tau) &= \int_{-\infty}^{\infty} h_1(t,\sigma)h_2(\sigma,\tau)d\sigma \\
&= \int_{-\infty}^{\infty} \sigma U(t-\sigma)\left[U(\sigma-\tau-1)-U(\sigma-\tau-2)\right]d\sigma \\
&= \int_{-\infty}^{\infty} \sigma U(t-\sigma)U(\sigma-\tau-1)d\sigma - \int_{-\infty}^{\infty} \sigma U(t-\sigma)U(\sigma-\tau-2)d\sigma \\
&= \left(\int_{\tau+1}^{t} \sigma d\sigma\right) U(t-\tau-1) - \left(\int_{\tau+2}^{t} \sigma d\sigma\right) U(t-\tau-2) \\
&= \frac{1}{2}[t^2-(\tau+1)^2]U(t-\tau-1) - \frac{1}{2}[t^2-(\tau+2)^2]U(t-\tau-2) \\
&= \begin{cases} 0, & \text{when } t-\tau<1, \\ \frac{1}{2}[t^2-(\tau+1)^2], & \text{when } 1<t-\tau<2, \\ \tau+\frac{3}{2}, & \text{when } t-\tau>2. \end{cases}
\end{aligned}
$$

(e) *Are $h_{1,2}(t,\tau)$ and $h_{2,1}(t,\tau)$ different, and if so why?*
They are different. The reason is that $\mathcal{S}_1$ is time-varying and so are $\mathcal{S}_{1,2}$ and $\mathcal{S}_{2,1}$.

(f) *Compute $H_2(s)$, the Laplace transform of $h_2(t)$, and find its region of convergence.*

$$
H_2(s) = \int_1^2 e^{-st}dt = (e^{-s}-e^{-2s})/s.
$$

To show that the region of convergence is the whole complex plane, consider

$$
H_2(0) = \int_1^2 dt = 1,
$$

which implies that $s=0$ is NOT a pole of $H_2(s)$.

3. *Compute the Laplace transform and corresponding region of convergence of the following functions. You are strongly advised to use the properties and the table of known Laplace transform pairs as much as possible.*

(a) $f(t) = tU(t-2)$.
$f(t) = (t-2)U(t-2) + 2U(t-2)$, therefore

$$
F(s) = e^{-2s}\left(-\frac{d}{ds}\frac{1}{s}\right) + 2e^{-2s}\frac{1}{s} = e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right).
$$

ROC is $\Re\{s\}>0$.

(b) $f(t) = \int_0^t \sigma \cos(t-\sigma) d\sigma$.
$f(t) = \int_{-\infty}^{\infty} g(\sigma)h(t-\sigma)d\sigma = g(t)*h(t)$, where $g(t) = tU(t)$ and $h(t) = U(t)\cos(t)$. $G(s) = \frac{1}{s^2}$ and $H(s) = \frac{s}{s^2+1}$, therefore $F(s) = G(s)H(s) = \frac{1}{s(s^2+1)}$. The ROC is $\Re\{s\} > 0$.

(c) $f(t) = \int_0^t e^{\sigma} \sin(\sigma) d\sigma$.
$f(t) = \int_{-\infty}^{t} g(\sigma)d\sigma$, where $g(t) = U(t)e^t \sin(t)$. $G(s) = \frac{1}{(s-1)^2+1}$ and $F(s) = G(s)/s = \frac{1}{s((s-1)^2+1)}$. The ROC is $\Re\{s\} > 1$.