EE102: Systems and Signals — Midterm Exam
Wednesday, July 16, 2014

This exam has 3 questions, for a total of 50 points.

Closed book. One two-sided cheat-sheet allowed.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
Please, write your name and ID on the top of each loose sheet!

Name and ID: ______________________________

Name of person on your left: ______________________________

Name of person on your right: ______________________________

Question	Points	Score
1	20	20
2	20	18
3	10	10
Total:	50	48

1. Consider the system described by the following differential equation:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} - \frac{\mathrm{e}^{-t}}{1+\mathrm{e}^{-t}}\, y(t) = x(t), \quad t > 0, \quad y(0) = 0$$

(a) (5 points) Find $h_\tau(t)$, the impulse response of the system above.

(b) (5 points) Is it a linear system?

(c) (5 points) Is it a time-invariant system?

(d) (5 points) Is it a causal system?

$y(t) = T[x(t)]$

(b) $T[k_1 x_1(t) + k_2 x_2(t)]$

$= \frac{1}{1+e^{-t}} \int_0^t [k_1 x_1(t) + k_2 x_2(t)](1+e^{-\sigma})\,d\sigma$

$= \frac{k_1}{1+e^{-t}} \int_0^t x_1(t)(1+e^{-\sigma})\,d\sigma + \frac{k_2}{1+e^{-t}} \int_0^t x_2(t)(1+e^{-\sigma})\,d\sigma$

$= k_1 T[x_1(t)] + k_2 T[x_2(t)]$ Linear system

(c) $h_\tau(t)$ · e^{-t} not a function of $(t-\tau)$ ∴ time varying system

(a) $(1+e^{-t})y' - e^{-t}y = x(1+e^{-t})$

$[(1+e^{-t})y]' = x(1+e^{-t})$

$(1+e^{-t})y(t)\Big|_0^t = \int_0^t x(\sigma)(1+e^{-\sigma})\,d\sigma$

$(1+e^{-t})y(t) - (1+e^{0})y(0) = \int_0^t x(\sigma)(1+e^{-\sigma})\,d\sigma$

$y(t) = \frac{1}{1+e^{-t}} \int_0^t x(\sigma)(1+e^{-\sigma})\,d\sigma$

$h_\tau(t) = \frac{1}{1+e^{-t}} \int_0^t \delta(\sigma-\tau)(1+e^{-\sigma})\,d\sigma$

$= \frac{1}{1+e^{-t}} \int_{-\infty}^{\infty} \delta(\sigma-\tau)(1+e^{-\sigma})u(\sigma)u(t-\sigma)\,d\sigma$

$= \frac{1}{1+e^{-t}}(1+e^{-\tau})u(\tau)u(t-\tau)$

$\tau > 0$, $t > \sigma$

(d) $y(t) = T[x(t)]$,

$y(t_0) = \frac{1}{1+e^{-t_0}} \int_0^{t_0} x(\sigma)(1+e^{-\sigma})\,d\sigma$

∴ integrated from 0 to t_0

∴ $y(t_0)$ is function of $x(t)$ for $t < t_0$ $\forall t_0$,

∴ Causal system

(C)

$$y(t-\tau) = \frac{1}{1+e^{-(t-\tau)}} \int_0^{t-\tau} x(\sigma)\,(1+e^{-\sigma})\,d\sigma$$

$z(t) = x(t-\tau)$

$$T[x(t-\tau)] = \frac{1}{1+e^{-t}} \int_0^t z(\sigma)\,(1+e^{-\sigma})\,d\sigma$$
$= T[z(t)]$

$$= \frac{1}{1+e^{-t}} \int_0^t x(\sigma-\tau)(1+e^{-\sigma})\,d\sigma$$

$\sigma - \tau = u$

$d\sigma = du$

$\sigma = u+\tau$

$$= \frac{1}{1+e^{-t}} \int_0^t x(u)\,(1+e^{-u-\tau})\,du$$

$y(t-\tau) \neq T[x(t-\tau)]$

$X(\beta t) = \frac{\delta(t)}{\beta}$

2. Consider the system described by the following differential equation:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} - 2y(t) = x(\beta t), \quad \beta > 0, \quad t > 0, \quad y(0) = 0$$

(a) (5 points) Find $h_\tau(t)$, the impulse response of the system.

(b) (5 points) Let $x(t) = \delta(t)$, an impulse function sitting at zero. Find $y(t)$ by applying the Laplace transform on the differential equation above and taking the inverse Laplace transform of the resulting $Y(s)$. Is the answer you got same as (a) with $\tau = 0+$?

(c) (5 points) Let $x(t) = \delta(t-\tau)$, an impulse function sitting at $\boxed{\tau}$. Find $y(t)$ by applying the Laplace transform on the differential equation above and taking the inverse Laplace transform of the resulting $Y(s)$. Is the answer you got same as (a)? (Assume $\tau > 0$)

(d) (5 points) Let $x(t) = \mathrm{u}(t-\tau)$. Can we find $y(t)$ by multiplying $X(s) = \mathcal{L}\{x(t)\}$ and $H(s) = \mathcal{L}_s\{h_\tau(t)\}$ and taking the inverse Laplace transform of $X(s)H(s)$? Why or why not? $x(t) * h(t) \longleftrightarrow X(s)\,H(s)$

(a) $\frac{dy(t)}{dt} - 2y(t) = x(\beta t)$

$(ye^{-2t})' = x(\beta t)e^{-2t}$

$\int_0^t (ye^{-2\sigma})' d\sigma = \int_0^t x(\beta\sigma')e^{-2\sigma'}d\sigma'$

$y(t)e^{-2t} - 0 = \int_0^t x(\beta\sigma')e^{-2\sigma'}d\sigma'$

$y(t) = e^{2t}\int_0^t x(\beta\sigma')e^{-2\sigma'}d\sigma'$

$h_\tau(t) = e^{2t}\int_0^t \frac{\delta(\sigma'-\tau)}{\beta}e^{-2\sigma'}d\sigma'$

$= \frac{e^{2t}}{\beta}\int_{-\infty}^{\infty} \delta(\sigma'-\tau)e^{-2\sigma'}u(\sigma')u(t-\sigma')d\sigma'$

$= \frac{e^{2t}}{\beta}e^{-2\tau}u(\tau)u(t-\tau)$

(b) $\frac{dy(t)}{dt} - 2y(t) = x(\beta t) = \frac{\delta(t)}{\beta}$ $\quad X(\beta t) = \delta(\beta t) = \frac{\delta(t)}{\beta}$

$sY(s) - y(0) - 2Y(s) = \frac{1}{\beta}$

$(s-2)Y(s) = \frac{1}{\beta}$

$Y(s) = \frac{1}{\beta(s-2)}$

$y(t) = \frac{1}{\beta}e^{2t}u(t)$

from (a) $h(t) = \frac{e^{2t}}{\beta}\cancel{e^{-2(0)}}u(t)$

$= \frac{e^{2t}}{\beta}u(t)$ — Yes

(c) $y_0(t) = \delta(t)$ $\quad y(t) = \delta(t-\tau)$

$sY(s) - y(0) - 2Y(s) = e^{-\tau s}$

Same $\frac{dy(t)}{dt} - 2y(t) = \frac{\delta(t-\tau)}{\beta}$

$sY(s) - 2Y(s) = \frac{e^{-\tau s}}{\beta}$

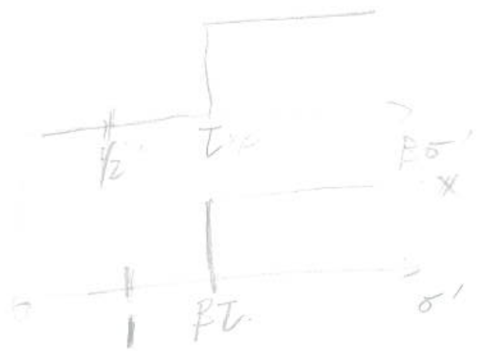
$Y(s) = \frac{e^{-\tau s}}{\beta(s-2)}$

$y(t) = \frac{1}{\beta}u(t-\tau)\, e^{2(t-\tau)}$

$Y(s) = \frac{1}{s-2}$, $y(t) = e^{2t}u(t)$, $y(t-\tau) = e^{2(t-\tau)}u(t-\tau)$

$y(t) = \frac{e^{2(t-\tau)}}{\beta}u(t-\tau)$ Yes

In $\beta\sigma'$ scale



(d)

$$x(t) = u(t-T)$$

$$y(t) = e^{2t}\int_0^t x(\beta\sigma')e^{-2\sigma'}d\sigma'$$

$$= e^{2t}\int_0^t u(\beta\sigma'-T)e^{-2\sigma'}d\sigma' \quad \cancel{e^{2t}\int_{\beta T}^t e^{-2\sigma'}d\sigma'}$$

$$h_T(t) = \frac{e^{2(t-T)}}{\beta}u(t)\ u(t-T)$$

$$y(t) = \int_{-\infty}^{\infty} h_T(t)\,x(T)\,dT$$

$$= \int_{-\infty}^{\infty}\frac{e^{2(t-T)}}{\beta}u(T)\,u(t-T)\,u(T-T)\,dT$$

$$= \int_0^t \frac{e^{2(t-T)}}{\beta}\,dT$$

$$= e^{2t}\int_0^t \frac{e^{-2T}}{\beta}\,dT$$

(d)

$\boxed{\text{No}}$

$y(t)$ is not a time-invariant system.

$$y(t) = e^{2t}\int_0^t x(\beta\sigma')e^{-2\sigma'}d\sigma'$$

$\cancel{x\beta\sigma}$

$x(t) = u(t-T)$

$x(\beta\sigma') = u(\beta\sigma'-T)$

$$= e^{2t}\int_0^t \underbrace{u(\beta\sigma'-T)}_{\text{Not a function of }(\sigma'-T)}e^{-2\sigma'}d\sigma'$$

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3. (10 points) Compute the inverse Laplace transform of $F(s)$, where

$$F(s) = \frac{s^2}{s^2 + 6s + 10}$$

$$F(s) = \frac{s^2}{s^2+6s+10} = \frac{s^2}{s^2+6s+3^2+1} = \frac{s^2}{(s+3)^2+1} = \frac{s^2+6s+10-6s-10}{s^2+6s+10}$$

$$= 1 - \frac{6s+10}{(s+3)^2+1} = 1 - \frac{6s+18-8}{(s+3)^2+1} = 1 - 6\frac{(s+3)}{(s+3)^2+1} + \frac{8}{(s+3)^2+1}$$

$$= 1 - 6\frac{s+3}{(s+3)^2+1^2} + 8\frac{1}{(s+3)^2+1^2}$$

$$f(t) = \left(\delta(t) - 6\cos(t)e^{-3t} + 8\sin(t)e^{-3t}\right)u(t)$$

$(s+3)^2$

$= (s+3)(s+3)$

$= s^2+9+6s$