This exam has 3 questions, for a total of 50 points.

Closed book. One two-sided cheat-sheet allowed.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
Please, write your name and ID on the top of ... !

Name and ID: ______

Name of person on ______

Name of person on ______

Question	Points	Score
1	20	19
2	20	13
3	10	9.5
Total:	50	41.5

1. Consider the system described by the following differential equation:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} - \frac{\mathrm{e}^{-t}}{1+\mathrm{e}^{-t}}\,y(t) = x(t), \quad t>0, \quad y(0)=0$$

(a) (5 points) Find $h_\tau(t)$, the impulse response of the system above.

(b) (5 points) Is it a linear system?

(c) (5 points) Is it a time-invariant system?

(d) (5 points) Is it a causal system?

$\int \frac{e^{-t}}{1+e^{-t}}\,dt$

a)

$$y' - \frac{e^{-t}}{1+e^{-t}}\,y = x$$

$$u = e^{-\int \frac{e^{-t}}{1+e^{-t}}dt} = e^{\ln(1+e^{-t})} = 1+e^{-t}$$

$$\frac{d}{dt}\left[y(1+e^{-t})\right] = x(t)(1+e^{-t})$$

$$y(1+e^{-t}) = \int_0^t x(t')dt' + \int_0^t x(t')e^{-t'}dt'$$

$$y(t) = \frac{1}{1+e^{-t}}\int_0^t x(\tau)d\tau + \frac{1}{1+e^{-t}}\int_0^t x(\tau)e^{-\tau}d\tau$$

$$\boxed{h_\tau(t) = \left[\frac{u(t-\tau)}{1+e^{-t}} + \frac{1}{1+e^{-t}}e^{-\tau}\,u(t-\tau)\right]u(\tau)} \quad -1$$

b) Yes it is a linear system. because integration is a linear operation as you can see here $y(k_1x_1+k_2x_2) = k_1y(x_1)+k_2y(x_2)$ ✓

c) it is NOT time invariant. It's time varying because $h_\tau(t)$ cannot be written as $h(t-\tau)$ since it does not only depend on $t-\tau$. ✓

d) Yes it is causal

because $h_\tau(t)=0$ for $t<\tau$ OK also because the upper limit of the integration is t, so the system does not depend on future input ✓

2. Consider the system described by the following differential equation:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} - 2y(t) = x(\beta t), \quad \beta > 0, \quad t > 0, \quad y(0) = 0$$

(a) (5 points) Find $h_\tau(t)$, the impulse response of the system.

(b) (5 points) Let $x(t) = \delta(t)$, an impulse function sitting at zero. Find $y(t)$ by applying the Laplace transform on the differential equation above and taking the inverse Laplace transform of the resulting $Y(s)$. Is the answer you got same as (a) with $\tau = 0+$?

(c) (5 points) Let $x(t) = \delta(t - \tau)$, an impulse function sitting at τ. Find $y(t)$ by applying the Laplace transform on the differential equation above and taking the inverse Laplace transform of the resulting $Y(s)$. Is the answer you got same as (a)? (Assume $\tau > 0$)

(d) (5 points) Let $x(t) = \mathrm{u}(t - \tau)$. Can we find $y(t)$ by multiplying $X(s) = \mathcal{L}\{x(t)\}$ and $H(s) = \mathcal{L}_s\{h_\tau(t)\}$ and taking the inverse Laplace transform of $X(s)H(s)$? Why or why not?

a) $y' - 2y = x(\beta t)$

$u = e^{-\int 2\,dt} = e^{-2t}$

$ye^{-2t} = \int_0^t x(\beta t') e^{-2t'} dt'$

$x(t) = \delta(\sigma - t)$

$y = \int_0^t x(\beta\tau) e^{-2(\tau - t)} d\tau$

$x(\beta t) = \delta(\sigma - \beta t)$

$h_\tau(t) = \int_0^t \delta(\sigma - \beta\tau) e^{-2(\tau - t)} d\tau$

$\sigma = \beta\tau$, $\tau = \frac{\sigma}{\beta}$

$\frac{1}{\beta} d\sigma = d\tau$

$u(t-\tau)$

$= \frac{1}{\beta} e^{-2(\frac{\sigma}{\beta} - t)} u(t - \frac{\sigma}{\beta})$

$= \frac{1}{\beta} e^{-2(\frac{\tau}{\beta} - t)} u(t - \frac{\tau}{\beta})$

b) $\mathcal{L}[y' - 2y = x(\beta t)]$

$\to \mathcal{L}[y - 2y = \delta(\beta t)]$

$sY - 2Y = 1$ $\quad \frac{1}{\beta}$

$Y = \frac{1}{s-2}$ $\quad Y = \frac{1}{\beta(s-2)}$

$y(t) = e^{+2t} u(t)$ $\quad y(t) =$

which is the same as $h_0(t) = e^{-2(0-t)} u(t-0)$

$= e^{2t} u(t)$

c)

$$x = \delta(t-\tau)$$

$$x(\beta t) = \delta(\beta t - \tau)$$

$$\mathcal{L}[\,y' - 2y = \delta(\beta t - \tau)\,]$$

$$sY - 2Y = \mathcal{L}[\delta(\beta t - \tau)]$$

$$\int_0^\infty e^{-st}\delta(\beta t - \tau)\,dt$$

$$u = \beta t - \tau$$

$$t = \frac{u+\tau}{\beta}$$

$$du = \beta\,dt$$

$$\int_0^\infty e^{-s\left(\frac{u+\tau}{\beta}\right)}\delta(u)\,du$$

$$sY - 2Y = e^{-s\frac{\tau}{\beta}}$$

$$Y_{(s)} = \frac{e^{-s\frac{\tau}{\beta}}}{s-2} = \frac{e^{-(s-2)\frac{\tau}{\beta}}}{(s-2)}e^{-2\frac{\tau}{\beta}}$$

$$y(t) = e^{2t}u\left(t - \frac{\tau}{\beta}\right)e^{-2\frac{\tau}{\beta}}$$

$$= e^{2\left(t - \frac{\tau}{\beta}\right)}u\left(t - \frac{\tau}{\beta}\right)$$

which is the same as

$$h_\tau(t) = e^{-2\left(\frac{\tau}{\beta} - t\right)}u\left(t - \frac{\tau}{\beta}\right)$$

$$= e^{2\left(t - \frac{\tau}{\beta}\right)}u\left(t - \frac{\tau}{\beta}\right)$$

d). -4

$$y = x(t) * h(t)$$

$$\mathcal{L}[\,y = x(t) * h(t)\,]$$

$$Y = \mathcal{L}[\,x(t) * h(t)\,]$$

$$Y = X_{(s)} \cdot H_{(s)}$$

by property on the board

$$\mathcal{L}^{-1}[\,Y = X_{(s)} \cdot H_{(s)}\,]$$

$$y(t) = \mathcal{L}^{-1}[\,X_{(s)} \cdot H_{(s)}\,]$$

therefore the method works!

3. (10 points) Compute the inverse Laplace transform of $F(s)$, where

$$F(s) = \frac{s^2}{s^2 + 6s + 10}$$

$$= 1 - \frac{6s}{s^2+6s+10} - \frac{10}{s^2+6s+10}$$

$$= 1 - \frac{6s}{(s+3)^2 + 10 - 9} - \frac{10}{(s+3)^2 + 1}$$

$$= 1 - 6\frac{s+3}{(s+3)^2+1} + 6\frac{3}{(s+3)^2+1} - \frac{10}{(s+3)^2+1}$$

$$\mathcal{L}^{-1}\left[F(s) = 1 - 6\frac{s+3}{(s+3)^2+1} + \frac{8 \times 1}{(s+3)^2+1}\right]$$

-0.5

$$\boxed{f(t) = \delta(t) - 6e^{-3t}\cos(t) + 8e^{-3t}\sin(t)\, u(t)}$$