This exam has 4 questions, for a total of 75 points.

> Closed book. No calculator. No cheat-sheet.
> Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
> **Please, write your name and ID on the top of each loose sheet!**

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me of person on your right: [redacted]

| Question | Points | Score |
|---|---|---|
| 1 | 12 | 12 |
| 2 | 12 | 12 |
| 3 | 33 | 18 |
| 4 | 18 | 18 |
| Total: | 75 | 60 |

## Multiple-choice questions – Check all the answers that apply.

1. (a) (3 points) Consider the system with input-output relation:

$$y(t) = T[x(t)] = x(\ln(t)), \quad t > 0.$$

$$x(t) \longrightarrow \boxed{\quad \mathcal{S} \quad} \longrightarrow y(t) = T[x(t)]$$

Is it

$y(t-\tau) = x(\ln(t-\tau))$ $\quad z(t) = x(t-\tau)$

$T[z(t)]$ $\quad = x(\ln t - \tau)$

$\ln(t) = a \quad t = e^a$

- ☑ linear?
- ☐ time-invariant?
- ☑ causal?

(b) (3 points) Consider the system with input-output relation;

$$y(t) = \int_{-\infty}^{\infty} \sqrt{x(\tau)}h(t-\tau)\,\mathrm{d}\tau, \quad \text{assuming that } h(t) = 0 \text{ when } t < 1.$$

$y(t) = \int_1^{\infty} \sqrt{x(\tau)}\, h(t-\tau) d\tau$

Is it

- ☐ linear?
- ☑ time-invariant?
- ☑ causal?

(c) (3 points) Consider the linear, time-invariant, causal system with ramp response given by:

$$g(t) = tu(t-1) + (1-\cos(t))u(t). \qquad \delta(t-1) + u(t)$$

What is the impulse response of this system?

$T[tu(t)] = tu(t-1) + (1-\cos(t))u(t)$

$T[u(t)] = t\delta(t-1) + u(t-1) + (1-\cos(t))\delta(t)$

$+ \sin(t)u(t)$

$= \delta(t-1) + u(t-1) + 0 + \sin(t)u(t)$

- ☐ $\delta(t-1) - \cos(t)u(t)$?
- ☐ $u(t-1) + \sin(t)u(t)$?
- ☑ $\delta(t-1) + \cos(t)u(t)$?

(d) (3 points) Consider the system described by the following differential equation:

$$\ln(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \frac{1}{t}y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}, \quad t > 1, y(1) = 0, x(1) = 2.$$

$T[\delta(t)]$

$= \delta'(t-1)$

$+ \delta(t-1) + \cancel{\sin(t)\delta(t)}$

$+ \cos u(t)$

Is this system

$\int_1^t \frac{d(\ln(t)y(t))}{dt} = \int_1^t \frac{dx(t)}{dt}$

- ☐ linear?
- ☐ time-invariant?
- ☑ causal?

$\ln(t)y(t) - \ln(1)y(1) = x(t) - x(1)$

$\ln(t)y(t) = x(t) - 2$

$y(t) = \frac{x(t)}{\ln(t)} - \frac{2}{\ln(t)}$

2. Consider the signal

$$x(t) = \sqrt{1-\cos(t)}\,(\mathrm{u}(t) - \mathrm{u}(t-\pi)) + \sqrt{2}\mathrm{u}(t-\pi).$$

(a) (5 points) Use the trigonometric identities to simplify $\sqrt{1-\cos(t)}$ so that you only have an expression of the form $k_1\sin(k_2 t)$ or $k_1\cos(k_2 t)$, where $k_1$ and $k_2$ are constants .

(b) (2 points) Plot the signal $x(t)$ as a function of $t$.

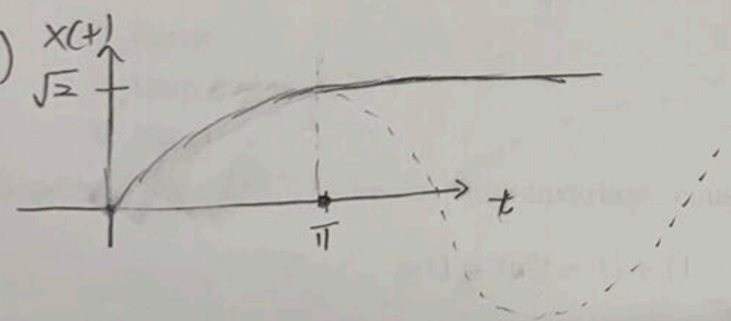
(c) (5 points) Compute the Laplace transform of $x(t)$ by using the tables. Remember to specify the region of convergence. (Hint: split up the signal into three signals and transform each of them. You do not need to combine the three transforms.)

a) $\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}$

$\sqrt{2}\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{1-\cos(\alpha)}$

$\sqrt{1-\cos(t)} = \sqrt{2}\sin\left(\frac{1}{2}t\right) \quad k_1 = \sqrt{2},\ k_2 = \frac{1}{2}$

b)




c) $x(t) = \sqrt{2}\sin\left(\frac{1}{2}t\right)u(t) - \sqrt{2}\sin\left(\frac{1}{2}t\right)u(t-\pi) + \sqrt{2}\,u(t-\pi)$

$\mathcal{L}\left[\sqrt{2}\sin\left(\frac{1}{2}t\right)u(t)\right] = \sqrt{2}\,\frac{\frac{1}{2}}{\frac{1}{4}+s^2} = \frac{\sqrt{2}}{\frac{1}{2}+2s^2} \quad Re(s) > 0$

$\mathcal{L}\left[-\sqrt{2}\sin\left(\frac{1}{2}t\right)u(t-\pi)\right] = \mathcal{L}\left[-\sqrt{2}\sin\left(\frac{1}{2}(t-\pi)+\frac{1}{2}\pi\right)u(t-\pi)\right]$

$= \mathcal{L}\left[\underbrace{\cancel{-\sqrt{2}\sin\left(\frac{1}{2}(t-\pi)\right)\cos\left(\frac{1}{2}\pi\right)u(t-\pi)}}_{0} - \sqrt{2}\cos\left(\frac{1}{2}(t-\pi)\right)\sin\left(\frac{1}{2}\pi\right)u(t-\pi)\right]$

$= \mathcal{L}\left[-\sqrt{2}\cos\left(\frac{1}{2}(t-\pi)\right)u(t-\pi)\right]$

$= -\sqrt{2}\,e^{-\pi s}\,\frac{s}{s^2+\frac{1}{4}} \quad Re\{s\} > 0$

$\mathcal{L}\left[\sqrt{2}\,u(t-\pi)\right] = \sqrt{2}\,\frac{e^{-\pi s}}{s} \quad Re\{s\} > 0$

3. Consider the series of two linear systems, $\mathcal{S}_1$ and $\mathcal{S}_2$. The first one has an input-output relationship given by:

$$y(t) = \int_{-\infty}^{t} e^{\sigma} x(\sigma)\, d\sigma,$$

and the second one has the input-output relationship given by:

$$z(t) = \int_{-\infty}^{t} e^{-(t-\sigma)} y(\sigma)\, d\sigma.$$

6 (a) (6 points) Check for linearity, time invariance, and causality for systems $\mathcal{S}_1$ and $\mathcal{S}_2$.

6 (b) (6 points) Determine the impulse response of systems $\mathcal{S}_1$ and system $\mathcal{S}_2$. Can you justify your answer in part (a) through the impulse response?

3 (c) (3 points) Let's assume that $\mathcal{S}_1$ receives the input $x(t) = e^{-t}u(t)$. What would be its output?

0 (d) (5 points) What would be the impulse response if we cascaded the two system such that $\mathcal{S}_2$ follows $\mathcal{S}_1$ (the output from $\mathcal{S}_1$ acts as an input for $\mathcal{S}_2$)? Check for linearity and time invariance of this system.

0 (e) (5 points) What would be the impulse response if we cascaded the two system such that $\mathcal{S}_1$ follows $\mathcal{S}_2$ (the output from $\mathcal{S}_2$ acts as an input for $\mathcal{S}_1$)? Check for linearity and time invariance of this system.

The system is LTV.

2 (f) (2 points) Do you expect the answers to parts (e) and (f) to be identical? Justify.

0 (g) (6 points) Compute the output for the case of $\mathcal{S}_2$ following $\mathcal{S}_1$ when $x(t) = e^{-t}u(t)$.

a) $S_1: T[\alpha_1 x_1(t) + \alpha_2 x_2(t)] = \int_{-\infty}^{t} e^{\sigma}(\alpha_1 x_1(\sigma) + \alpha_2 x_2(\sigma))\, d\sigma$

$= \int_{-\infty}^{t} e^{\sigma}\alpha_1 x_1(\sigma)\, d\sigma + \int_{-\infty}^{t} e^{\sigma}\alpha_2 x_2(\sigma)\, d\sigma$

$= \alpha_1 y_1(t) + \alpha_2 y_2(t)$ Linear

$y(t-\tau) = \int_{-\infty}^{t-\tau} e^{\sigma} x(\sigma)\, d\sigma$

$T[x(t-\tau)] = \int_{-\infty}^{t} e^{\sigma} x(\sigma-\tau)\, d\sigma \quad \sigma-\tau = u \quad d\sigma = du$

$\sigma = u+\tau$

$= \int_{-\infty}^{t-\tau} e^{u+\tau} x(u)\, du \neq y(t-\tau)$ tim-varying

Causal because integral evaluates from $-\infty$ to $t$.

$S_2: T[\alpha_1 y_1(t) + \alpha_2 y_2(t)] = \int_{-\infty}^{t} e^{-(t-\sigma)} \alpha_1 y_1(\sigma)\, d\sigma + \int_{-\infty}^{t} e^{-(t-\sigma)} \alpha_2 y_2(\sigma)\, d\sigma$

$= \alpha_1 z_1(t) + \alpha_2 z_2(t)$ Linear.

$$z(t-\tau) = \int_{-\infty}^{t-\tau} e^{-(t-\tau-\sigma)} y(\sigma)\,d\sigma$$

$$T[y(t-\tau)] = \int_{-\infty}^{t} e^{-(t-\sigma)} y(\sigma-\tau)\,d\sigma$$

$\sigma - \tau = u \quad d\sigma = du$

$\sigma = u + \tau$

$$= \int_{-\infty}^{t-\tau} e^{-(t-(u+\tau))} y(u)\,du$$

$$= \int_{-\infty}^{t-\tau} e^{-(t-\tau-u)} y(u)\,du = z(t-\tau) \text{ Time-Invariant}$$

Causal because integral evaluates from $-\infty$ to $t$ and $e^{-(t-\sigma)}$ where $t-\sigma \geq 0 \therefore t \geq \sigma$

b) $S_1$: $h_1(t;\tau) = \int_{-\infty}^{t} e^{\sigma}\delta(\sigma-\tau)\,d\sigma = \int_{-\infty}^{\infty} e^{\sigma} u(t-\sigma)\delta(\sigma-\tau)\,d\sigma$

$h_1(t;\tau) = e^{\tau}u(t-\tau)$

$e^{\tau}$ makes $S_1$ time-varying, $u(t-\tau)$ shows $S_1$ is causal.

$S_2$: $h_2(t;\tau) = \int_{-\infty}^{t} e^{-(t-\sigma)}\delta(\sigma-\tau)\,d\sigma = \int_{-\infty}^{\infty} e^{-(t-\sigma)} u(t-\sigma)\delta(\sigma-\tau)\,d\sigma$

$h_2(t;\tau) = e^{-(t-\tau)}u(t-\tau)$

since it's only a function of $(t-\tau)$, it's time-invariant.

$h_2(t;\tau) = e^{-t}u(t)$

$h_2(t;\tau) = 0$ for $t<0$ shows $S_2$ is causal.

c). $y(t) = \int_{-\infty}^{t} e^{\sigma}e^{-\sigma}u(\sigma)\,d\sigma = \int_{0}^{t} e^{\sigma-\sigma}\,d\sigma = t, \ t>0$

for $t<0$ $y(t)=0$ $\therefore y(t) = tu(t)$

d) $h_{12}(t;\tau) = \int_{-\infty}^{t} e^{-(t-\sigma)} h_1(t;\sigma)\,d\sigma = \int_{-\infty}^{t} e^{-(t-\sigma)} e^{\sigma}u(t-\sigma)\,d\sigma = \int_{-\infty}^{t} e^{-t+2\sigma}\,d\sigma = e^{-t}\left[\frac{1}{2}e^{2\sigma}\right]_{-\infty}^{t}$

$= e^{-t}\frac{1}{2}e^{2t} = \frac{1}{2}e^{t}, \ t>0$ Linear and time-varying.

e) $h_{21}(t;\tau) = \int_{-\infty}^{t} e^{\sigma}h_2(t;\sigma)\,d\sigma = \int_{-\infty}^{t} e^{\sigma}e^{-(t-\sigma)}u(t-\sigma)\,d\sigma = \int_{-\infty}^{t} e^{-t+2\sigma}\,d\sigma$

$= e^{-t}\left[\frac{1}{2}e^{2\sigma}\right]_{-\infty}^{t} = e^{-t}\frac{1}{2}e^{2t} = \frac{1}{2}e^{t} \ t>0$ linear and time-varying.

Checking for linearity:

If both $S_1$ and $S_2$ are linear, the output of $S_1$ and $S_2$ are also linear. Therefore, $S_{12}$ is linear.

Checking for time-invariance:

$\frac{1}{2}e^t$ makes it time varying as it's a function of time.

e) Checking for linearity:

Linear. Same reasoning as part (d)

Checking for time-invariance:

$e^{-t}$ is a function of time therefore $S_{21}$ is time-varying.

f) $\boxed{\text{No}}$ because $S_1$ is time varying.

$$\cancel{\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau}$$

In this case,

$$\cancel{\int_{-\infty}^{\infty} \underbrace{h_1(\tau)h_2(t-\tau)}_{\text{part (d)}}d\tau = \int_{-\infty}^{\infty} \underbrace{h_1(t-\tau)h_2(\tau)}_{\text{part (e)}}d\tau}$$

g) $S_{21}(t) = \int_{-\infty}^{t} h_{21}(t;\sigma)x(\sigma)d\sigma$

$= \int_{-\infty}^{t} \frac{1}{2}e^t e^{-\sigma}u(\sigma)d\sigma$

$= \frac{1}{2}e^t \int_0^t e^{-\sigma}d\sigma = \frac{1}{2}e^t[-e^{-\sigma}]_0^t = \frac{1}{2}e^t(-e^{-t}+1)$

$= \frac{1}{2}e^t - \frac{1}{2}$ for $t>0$

$S_{21}(t) = 0$ for $t<0$

4. Let

$$h(t) = e^{-t}u(t)$$

be the impulse response of a linear system, $\mathcal{S}$, and let $x(t)$ be defined as follows:

$$x(t) = \begin{cases} 0, & t < 0, \\ t, & 0 < t < 1, \\ -(t-2), & 1 < t < 2, \\ 0, & t > 2. \end{cases}$$

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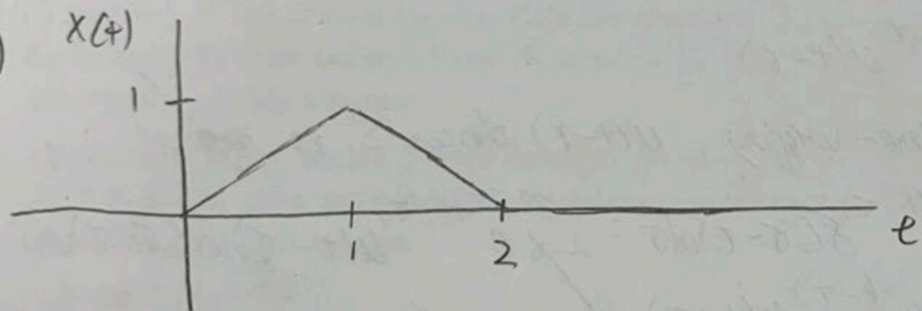
(a) (10 points) Sketch $x(t)$ and express it as a superposition of delayed and scaled replicas of $r(t) = tu(t)$, the ramp function. Your solution should not contain any function other than delayed and scaled versions of $r(t)$.

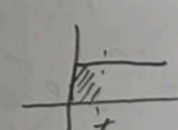
(b) (4 points) Compute the ramp response of $\mathcal{S}$.

(c) (4 points) Compute the response of the system when the input is $x(t)$.

a)




$$x(t) = tu(t) - tu(t-1) - (t-2)u(t-1) - (t-2)u(t-2)$$
$$= r(t) - r(t-2) - u(t-1)(t+t-2)$$
$$= r(t) - r(t-2) - u(t-1)(2t-2)$$
$$= r(t) - r(t-2) - u(t-1)2(t-1)$$
$$= r(t) + r(t-2) - 2r(t-1)$$



b) $T[\delta(t)] = h(t) = e^{-t}u(t)$

$$T[u(t)] = \int_{-\infty}^{t} h(\tau)\,d\tau = \int_{-\infty}^{t} e^{-\tau}u(\tau)\,d\tau = \int_0^t e^{-\tau}\,d\tau = (-e^{-t}+1)u(t)$$

$$T[r(t)] = T[tu(t)] = \int_{-\infty}^{t} T[u(\tau)]\,d\tau = \int_{-\infty}^{t} (-e^{-\tau}+1)u(\tau)\,d\tau$$
$$= \int_0^t (-e^{-\tau}+1)\,d\tau = [e^{\tau}+\tau]_0^t = (e^{t}+t-1)u(t)$$

c) $T[x(t)] = T[r(t)] - T[r(t-2)] - 2T[r(t-1)]$

$= (e^{t} + t - 1)u(t) - (e^{t-2} + t - 3)u(t-2) - 2(e^{t-1} + t - 2)u(t-1)$