UCLA — Electrical Engineering Dept.
EE102: Systems and Signals
Midterm Exam
Tuesday, May 8, 2007
Please, write your name and ID on the top of each sheet!

This exam has 3 questions, for a total of 16 points.

Closed book. No calculator. One two-sided cheat-sheet allowed.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Name and ID: [redacted]

	Points
#1	7
#2	5
#3	4
Total	16

 Please go to next page...

1. Consider the system described by the following input-output relationship

$$y(t) = x(t) + \int_{-\infty}^{t} x(\sigma)(t-\sigma)e^{-(t-\sigma)}\, d\sigma.$$

(a) (3 points) Is this system linear? Time-invariant? Causal? Briefly explain your answers.

(b) (1 point) Compute the impulse response of the system, $h(t)$.

(c) (1 point) Compute the Laplace transform of $h(t)$, $H(s)$.

(d) (1 point) Let $x(t) = e^{-(t-1)}U(t)$. Compute the Laplace transform of $x(t)$, $X(s)$.

(e) (1 point) Compute the Laplace transform of the output of the system, $Y(s) = \mathcal{L}\{y(t)\}$, when the input, $x(t)$, is the signal of the previous part.

a) The system is $\boxed{\text{linear.}}$

$k_1 x_1(t) + k_2 x_2(t) + \int_{-\infty}^{t} k_1 x_1(\sigma)(t-\sigma) e^{-(t-\sigma)} + \int_{-\infty}^{t} k_2 x_2(\sigma)(t-\sigma) e^{-(t-\sigma)} d\sigma$

$= k_1 y_1(t) + k_2 y_2(t)$

$T[x(t-\tau)] = x(t-\tau) + \int_{-\infty}^{t} x(\sigma-\tau)(t-\sigma) e^{-(t-\sigma)} d\sigma$

$y(t-\tau) = x(t-\tau) + \int_{-\infty}^{t-\tau} x(\sigma)(t-\tau-\sigma) e^{-(t-\tau-\sigma)} d\sigma$ $=$ $\boxed{\text{Time invariant}}$

Let $\tau' = \tau + \sigma \quad d\tau' = d\sigma$

$\tau' = \tau + t - \tau = t$

$x(t-\tau) + \int_{-\infty}^{t} x(\tau'-\tau)(t-\tau') e^{-(t-\tau')} d\tau'$

System is $\boxed{\text{causal}}$, only depends upto present. t

b) $h(t,\tau) = \delta(t-\tau) + \int_{-\infty}^{t} \delta(t-\tau)(t-\sigma) e^{-(t-\sigma)} d\sigma$

$= \delta(t-\tau) + \int_{-\infty}^{\infty} U(t-\sigma)(t-\sigma) e^{-(t-\sigma)} \delta(t-\tau) d\sigma$

$= \boxed{\delta(t-\tau) + U(t-\tau)(t-\tau) e^{-(t-\tau)}}$

$h(t) = ?$

c) $h(t) = \delta(t) + u(t)\, te^{-t}$

$$\boxed{H(s) = 1 + \frac{1}{(s+1)^2}}$$

$s+1=0$

$p_1: s=-1$

$$\boxed{ROC: Re\{s\} > -1}$$

d) $x(t) = e^{-(t-1)} u(t)$

$= e^{+1} e^{-t} u(t)$

$$X(s) = e \frac{1}{s+1} = \boxed{\frac{e}{s+1}}$$

$p_1: s=-1$ $\boxed{ROC: Re\{s\} > -1}$

e) $Y(s) = X(s)\,H(s)$

$$Y(s) = \left(\frac{e}{s+1}\right)\left(1 + \frac{1}{(s+1)^2}\right)$$

$$\boxed{Y(s) = \frac{e}{s+1} + \frac{e}{(s+1)^3}}$$

$p_1: s=-1$

$$\boxed{ROC: Re\{s\} > -1}$$

ROC!!

2. Let $x(t) = (e^{2t} - e^{-2t})U(t)$ be the input to a linear, time-invariant system with impulse response $h(t) = t\cos(2t)U(t)$.

   (a) (1 point) Compute $X(s) = \mathcal{L}\{x(t)\}$.

   (b) (1 point) compute $H(s) = \mathcal{L}\{h(t)\}$.

   (c) (1 point) Let $y(t)$ be the corresponding output of the system. Compute $Y(s) = \mathcal{L}\{y(t)\}$.

   (d) (2 points) Now let $y(t)$ be the input to a second system, whose input-output relationship is given by $z(t) = \int_0^t e^{-2\sigma} y(\sigma)\, d\sigma$. Compute $Z(s) = \mathcal{L}\{z(t)\}$.

a) $x(t) = (e^{2t} - e^{-2t})U(t)$

$X(s) = L\{e^{2t}U(t)\} - L\{e^{-2t}U(t)\}$ $p_1: s=2$, $p_2: s=-2$

$= \boxed{\frac{1}{s-2} - \frac{1}{s+2}}$ $\boxed{ROC: Re\{s\} > 2}$ ✓

b) $h(t) = t\cos(2t)U(t)$

$H(s) = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$ $p_{1,2}: s^2 = -4$, $s = \pm 2i$

$= \frac{-(s^2+4) + s(2s)}{(s^2+4)^2}$ $\boxed{ROC: Re\{s\} > 0}$

$= \boxed{\frac{s^2-4}{(s^2+4)^2}}$ ✓

c) $Y(s) = H(s)\,X(s)$

$= \frac{(s+2)(s-2)}{(s^2+4)^2}\left(\frac{1}{s-2} - \frac{1}{s+2}\right)$

$= \frac{\cancel{-}\ s+2}{(s^2+4)^2} - \frac{(s-2)}{(s^2+4)^2}$

$= \boxed{\frac{4}{(s^2+4)^2}}$ $\boxed{ROC: Re\{s\} > 0}$ ✓

d) $y(t) \rightarrow \boxed{S_2} \rightarrow z(t)$

$$z(t) = \int_0^t e^{-2\sigma} y(\sigma)\,d\sigma$$

$$z(t) = \int_{-\infty}^{\infty} u(\sigma)\,u(t-\sigma)\,e^{-2\sigma}(y(\sigma))\,d\sigma$$

$$h(t) = u(t)$$

$$g(t) = u(t)\,e^{-2t}\,y(t)$$

$$Z(S) = H(S)G(S)$$

$$H(S) = \frac{1}{S}$$

$$G(S) = Y(S+2) = \frac{4}{[(S+2)^2+4]^2}$$

$$\boxed{Z(S) = \frac{4}{S[(S+2)^2+4]^2}}$$

$p_1 = 0$

$(s+2)^2 + 4 = 0$

$s^2 + 4s + 4 + 4 = 0$

$s^2 + 4s + 8 = 0$

$$s = \frac{-4 \pm \sqrt{16 - 4(1)(8)}}{2}$$ $Re\{s\}$

$\boxed{\text{ROC: } Re\{S\} > 0}$

✓

$U(t) \to \boxed{LTI} \to (t+1)e^{-t}U(t)$

$\delta(t-1) - U(t-2) \to \boxed{LTI} \to ?$

$= \int \delta(t-1)h(t)dt - \int U(t-2)\,h(t)\,dt$

$h(t-1) \qquad = (t-1)e^{-(t-2)}U(t-2)$

3. (a) (2 points) Let $g(t) = (t+1)e^{-t}U(t)$ be the *step response* of a given linear time-invariant system. Compute the response of the same system to $x(t) = \delta(t-1) - U(t-2)$.

   (b) (2 points) Consider a linear time-invariant system (different from the one in the previous part). Of this system we only know that the response to $x_1(t) = e^{-t}U(t-1)$ is $y_1(t) = (t-1)e^{-t}U(t-1)$ and the response to $x_2(t) = \delta(t-1) + e^{-t}U(t)$ is $y_2(t) = e^{-t}[tU(t) + eU(t-1)]$. Can you compute the system's impulse response, $h(t)$? If so, what is it?

a) $g(t) = (t+1)e^{-t}U(t)$ LTI

step response

$$y(t) = \int \delta(t-1)h(t)\,dt - \int U(t-2)\,h(t)\,dt$$

$\boxed{T.I.}$

$h(t-1) \qquad g(t-2) = (t-1)e^{-(t-2)}U(t-2)$

$g(t) = \int h(t)\,U(t)\,dt$

$h(t) = \dfrac{dg(t)}{dt}$

$$\frac{dg}{dt} = [e^{-t} - (t+1)e^{-t}]U(t) + (t+1)e^{-t}\delta(t)$$

$$h(t) = e^{-t}U(t) - (t+1)e^{-t}U(t) + \delta(t)$$

$$h(t-1) = e^{-(t-1)}U(t-1) - te^{-(t-1)}U(t-1) + \delta(t-1)$$

$$\boxed{y(t) = [e^{-(t-1)}U(t-1)][1-t] + \delta(t-1) - (t-1)e^{-(t-2)}U(t-2)}$$

b) LT

$$e^{-t}u(t-1) \to \boxed{LTI} \to (t-1)e^{-t}u(t-1)$$

$$\delta(t-1) + e^{-t}u(t) \Rightarrow \boxed{LTI} \to e^{-t}[tu(t) + eu(t-1)]$$

Since $Y(s) = X(s)H(s)$

then $H(s) = \frac{Y(s)}{X(s)}$

$$h(t) = L^{-1}\left\{\frac{Y(s)}{X(s)}\right\}$$

$$X(s) = e^{-1}L\{e^{-(t-1)}u(t-1)\}$$

$$L\{e^{-t}u(t)\} = \frac{1}{s+1}$$

$$X(s) = e^{-1}\left[e^{-s}\left(\frac{1}{s+1}\right)\right]$$

$$L\{te^{-t}\}$$

$$Y(s) = e^{-1}L\{e^{-(t-1)}(t-1)u(t-1)\}$$

$$= e^{-1}\left[e^{-s}\left(\frac{1}{(s+1)^2}\right)\right]$$

$$\frac{Y(s)}{X(s)} = \frac{\cancel{e^{-1}}\,\cancel{e^{-s}}\left(\frac{1}{(s+1)^{\cancel{2}}}\right)}{\cancel{e^{-1}}\left[\cancel{e^{-s}}\left(\frac{1}{\cancel{s+1}}\right)\right]}$$

$$H(s) = \frac{1}{s+1}$$

$$L^{-1}\{H(s)\} = \boxed{h(t) = e^{-t}u(t)}$$