UCLA — Electrical Engineering Dept.
EE102: Systems and Signals
Midterm Solutions

1. *Consider the following systems.*

   (a) $\mathcal{S}_1$*:* $y(t) = \sin(x(t))$*. Is it linear? Time-invariant? Causal? Explain.*
   Nonlinear: $\sin(k_1x_1(t) + k_2x_2(t)) \neq k_1 \sin(x_1(t)) + k_2 \sin(x_2(t))$. Time-invariant: $\sin(x(t-\tau)) = y(t-\tau)$. Causal and memoryless: $y(t_0)$ depends only on $x(t_0)$.

   (b) $\mathcal{S}_2$*:* $y(t) = \int_{t-2}^{t}(t-\sigma)x(\sigma)d\sigma$*. Is it linear? Time-invariant? Causal? Explain.*
   One can write $y(t) = \int_{-\infty}^{\infty}[U(\sigma - t + 2)U(t-\sigma)](t-\sigma)x(\sigma)d\sigma = x(t) * h(t)$, with $h(t) = [U(-t+2)U(t)]t$. Note that $h(t)$ can be also written as $h(t) = [U(t) - U(t-2)]t$. The system is then a linear, time-invariant system with impulse response $h(t)$. Because $h(t) = 0$, $t < 0$, the system is also causal.

   (c) *Consider now the* ***time-invariant*** *system* $\mathcal{S}_3$*, and assume that the responses to the two signals* $x_1(t) = U(t)$ *and* $x_2(t) = U(t) - U(t-2)$ *are respectively* $y_1(t) = U(t-1)$ *and* $y_2(t) = U(t-1) - U(t-2)$*. Is* $\mathcal{S}_3$ *linear?*
   If the system were linear, the output corresponding to $x_3(t) = x_1(t) - x_2(t)$ would be equal to $y_3(t) = y_1(t) - y_2(t)$. We have that $x_3(t) = U(t-2)$ and $y_3(t) = U(t-2)$. This contradicts the fact that the system is time-invariant: $x_3(t)$ is simply equal to $x_1(t-2)$, therefore the corresponding output should be equal to $y_1(t-2) = U(t-3)$. [If you are curious to know what the input-output relation for this system is, here it is: $y(t) = x(t)x(t-1)$.]

2. *Consider the system* $\mathcal{S}_2$ *from the previous question.*

   (a) *Compute its impulse response,* $h(t)$*.*
   See above: $h(t) = t[U(t) - U(t-2)]$.

   (b) *Compute the output of the system when* $x(t) = U(t-1)$*. Use the convolution integral.* Because the system is time-invariant, one can compute the output corresponding to $\bar{x}(t) = x(t+1) = U(t)$ and then shift it to the right by 1. $\bar{y}(t) = \int_{t-2}^{t}(t-\sigma)\bar{x}(\sigma)d\sigma = \int_{t-2}^{t}(t-\sigma)U(\sigma)d\sigma$. If $t < 0$, $\bar{y}(t) = 0$. If $0 < t < 2$, then $\bar{y}(t) = \int_0^t (t-\sigma)d\sigma = [t\sigma - \sigma^2/2]_{\sigma=0}^{t} = t^2/2$. If $t > 2$, then $\bar{y}(t) = \int_{t-2}^{t}(t-\sigma)d\sigma = [t\sigma - \sigma^2/2]_{\sigma=t-2}^{t} = 2$. The desired output $y(t)$ is equal to $y(t) = \bar{y}(t-1)$.

3. *Consider now the system* $\mathcal{S}_4$*, described by the following input-output relation:*

   $$\mathcal{S}_4 : y(t) = t^2x(t).$$

   (a) *Suppose that the output of system* $\mathcal{S}_2$ *above is sent as input to system* $\mathcal{S}_4$*,*

   $$\overset{a(t)}{\longrightarrow} \boxed{\mathcal{S}_2} \overset{b(t)}{\longrightarrow} \boxed{\mathcal{S}_4} \overset{c(t)}{\longrightarrow}.$$

   What is the output of the overall system when $a(t) = \delta(t-1)$*?*
   The output of $\mathcal{S}_2$ is equal to $b(t) = h(t-1) = (t-1)[U(t-1) - U(t-3)]$, because the system is linear, time-invariant. Therefore $c(t) = t^2(t-1)[U(t-1) - U(t-3)]$.

(b) *The two systems are now switched:*

$$\stackrel{a(t)}{\longrightarrow} \boxed{\mathcal{S}_4} \stackrel{b(t)}{\longrightarrow} \boxed{\mathcal{S}_2} \stackrel{c(t)}{\longrightarrow}.$$

What is the output of this combined system when $a(t) = \delta(t-1)$? The output of $\mathcal{S}_4$ is now equal to $b(t) = t^2\delta(t-1) = \delta(t-1)$. Therefore $c(t) = h(t-1) = (t-1)[U(t-1) - U(t-3)]$.

4. *Compute the Laplace transform of the following signals:*

   (a) $f_1(t) = e^{-t}\cos(t-1)U(t-1)$;
   Because $f_1(t) = e^{-1}g(t-1)$, where $g(t) = e^{-t}\cos(t)U(t)$, then $F_1(s) = e^{-1}e^{-s}G(s) = e^{-(s+1)}(s+1)/[(s+1)^2+1]$, with a pole in $s = -1$, therefore the region of convergence is given by $\Re\{s\} > -1$.

   (b) $f_2(t) = \int_0^t (t-\sigma)e^{-\sigma}d\sigma$;
   Re-write $f_2(t)$ as $f_2(t) = \int_{-\infty}^{\infty}[(t-\sigma)U(t-\sigma)][U(\sigma)e^{-\sigma}]d\sigma = [tU(t)] * [U(t)e^{-t}]$, then $F_2(s) = (1/s^2)[1/(s+1)] = 1/[s^2(s+1)]$, with poles in $s = 0$ and $s = -1$, therefore the ROC is $\Re\{s\} > 0$.

   (c) $f_3(t) = \int_0^t \sin(\sigma)\cos(\sigma)d\sigma$.
   Re-write $f_3(t)$ as $f_3(t) = \int_{-\infty}^{\infty}[(1/2)\sin(2\sigma)U(\sigma)][U(t-\sigma)]d\sigma = [(1/2)U(t)\sin(2t)] * U(t)$, then $F_3(s) = (1/2)[2/(s^2+4)](1/s) = 1/[s(s^2+4)]$ with poles in $s = 0$, $s = \pm 2i$, therefore the ROC is $\Re\{s\} > 0$.