UCLA
Dept. of Electrical Engineering
EE102: Systems and Signals
Midterm Exam
Thursday, May 12, 2005
Please, write your name and ID on the top of each sheet!

① 10
② 10
③ 10
④ 20
⑤ 9
59

This exam has 5 questions, for a total of 60 points.

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Name and ID: ______________________

Name of person to your left: ______________________

Name of person to your right: ______________________

1. [10 points] Which of the following systems are linear, time-invariant, causal, and why?

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(a) $y(t) = x(t+2) - x(t-2)$; linear

(b) $y(t) = x(t)x(t-1)$; non linear

(c) $y(t) = \int_{-\infty}^{t+1} x(\sigma)d\sigma$. non-causal

$y(t-\tau) = \int_{-\infty}^{t-\tau+1} x(\sigma)\,d\sigma$

$T[x(t-\tau)] = \int_{-\infty}^{t+1}$

a) $T[x_1(t)] = x_1(t+2) - x_1(t-2)$

$T[x_2(t)] = x_2(t+2) - x_2(t-2)$

$T[k_1x_1(t) + k_2x_2(t)] = k_1x_1(t+2) + k_2x_2(t+2) - [k_1x_1(t-2) + k_2x_2(t-2)]$

$= k_1[x_1(t+2) - x_1(t-2)] + k_2[x_2(t+2) - x_2(t-2)]$

$= k_1 T[x_1(t)] + k_2 T(x_2(t)]$ $\boxed{\text{linear}}$

$y(t-\tau) = x(t-\tau+2) - x(t-\tau-2)$

$T[x(t-\tau)] = x(t-\tau+2) - x(t-\tau-2)$ $\boxed{TI}$

$y(t_0) = x(t_0+2) - x(t_0-2)$

$\boxed{\text{non-causal}}$ — need to know future to evaluate

b) $\boxed{\text{non-linear}}$ since input is "squared"

$y(t-\tau) = x(t-\tau)\,x(t-\tau-1)$

$T[x(t-\tau)] = x(t-\tau)x(t-\tau-1)$ $\boxed{TI}$

$y(t_0) = x(t_0)x(t_0-1)$ $\boxed{\text{causal}}$ don't need future times

c) $\boxed{\text{linear}}$ b/c integral is linear operator

$$y(t) = \int_{-\infty}^{t+1} x(\sigma)\,d\sigma$$

$$y(t-\tau) = \int_{-\infty}^{t-\tau+1} x(\sigma)\,d\sigma$$

$$T[x(t-\tau)] = \int_{-\infty}^{t+1} x(\sigma-\tau)\,d\sigma$$

$u = \sigma - \tau$
$du = d\sigma$

$$= \int_{-\infty}^{t+1-\tau} x(u)\,du \Rightarrow \int_{-\infty}^{t-\tau+1} x(\sigma)\,d\sigma$$

$\therefore \boxed{TI}$

$\boxed{\text{non-causal}}$ b/c need future $\int_{-\infty}^{t+1} x(\sigma)\,d\sigma$

10 2. $\boxed{\text{10 points}}$ Find the inverse Laplace transform of the function

$$F(s) = \frac{e^{-(s-1)} + 3}{s^2 - 2s + 5}, \quad \Re\{s\} > 1. \tag{1}$$

$$F(s) = \frac{e^{-(s-1)}}{s^2-2s+5} + \frac{3}{s^2-2s+5}$$

$$e^{-s+1}$$

$$= e\left(\frac{e^{-s}}{s^2-2s+5}\right) + \frac{3}{s^2-2s+5}$$

$$\frac{ee^{-s}}{s}$$

↑ delayed signal $f(t-1)u(t-1)$

find $\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\}$ first

$$\frac{1}{s^2-2s+5} = \frac{1}{(s-1)^2+4} \cdot \frac{2}{2} = \frac{1}{2} \cdot \frac{2}{(s-1)^2+4} \quad \left(" \frac{\omega}{s^2+\omega^2} " \right)$$

$$\mathcal{L}^{-1}\{\cdot\} \Rightarrow \frac{1}{2}\sin(2t)u(t)e^t \qquad \uparrow e^t$$

$$\Rightarrow \therefore e\left(\frac{e^{-s}}{s^2-2s+5}\right) + \frac{3}{s^2-2s+5} = \boxed{e\left(\frac{1}{2}\sin 2(t-1)u(t-1)e^{t-1}\right) + \frac{3}{2}\sin 2t\, u(t)e^t}$$

$$\parallel$$

$$\mathcal{L}^{-1}\{F(s)\}$$

2 & 3 possible.

3. $\boxed{\text{10 points}}$ The linear, time-invariant, causal system S admits the output $y(t) = \frac{1}{9}(1 - \cos(3t))U(t)$ when the input is $x(t) = tU(t)$. What is the output $y_1(t)$ when the input is $x_1(t) = U(t)$? Do not use the Laplace transform.

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$x(t) = tu(t) \xrightarrow{S} \boxed{LTI, C} \longrightarrow y(t) = \frac{1}{9}(1 - \cos(3t))u(t) = \frac{1}{9}u(t) - \frac{\cos(3t)}{9}u(t)$

$x_1(t) = u(t) \xrightarrow{S} \boxed{LTI, C} \longrightarrow ?\ y_1(t)$

$\frac{1}{9}(u(t) - u(t)\cos 3t)$

$\frac{1}{9}(\cancel{\delta(t) - \delta(t)\cos 3t}$

$+3u(t)\sin 3t$

$\frac{3u(t)\sin 3t}{9} = u(t)\frac{\sin 3t}{3}$

note: $\frac{d}{dt}tu(t) = u(t) + \underbrace{\cancel{t\delta(t)}}_{\text{sifting}}$

$\frac{d}{dt}tu(t) = u(t)$

$\frac{d}{dt}T[tu(t)] = T\left[\frac{d}{dt}tu(t)\right] = T[u(t)] = \frac{d}{dt}T[tu(t)]$

$= \frac{d}{dt}y(t)$

$y_1(t) = T[u(t)] = \left[\frac{1}{9}u(t) - \frac{\cos 3t}{9}u(t)\right]'$

$= \frac{\delta(t)}{9} - \left(\frac{1}{9}\left[\underset{\cos 3(0) = 1}{\cos 3t}\,\delta(t) - 3\sin 3t\, u(t)\right]\right)$

$= \frac{\delta(t)}{9} - \left(\frac{\delta(t)}{9} - \frac{\sin 3t\, u(t)}{3}\right)$

$\boxed{y_1(t) = \frac{\sin 3t\, u(t)}{3}}$

$\cancel{+\delta(t)}$ $\cancel{u(t)}$

$\cancel{\frac{1}{9}(u(t) - \cos 3t\, u(t))}$

$\cancel{= \frac{1}{9}(\delta(t) - \cos 3t\,\delta(t) + 3\sin 3t\, u(t))}$

$\cancel{\delta(t) - \delta(t)} + 3\sin 3t\, u(t)$

$\frac{\sin 3t\, u(t)}{3}$

L, TI, C.

4. When $x_1(t) = e^{-(t-1)}U(t-1)$ is the input to linear, time-invariant, causal system, $\mathcal{S}_1$, the corresponding output is $y_1(t) = e^{-(t-1)}\sin(t-1)U(t-1)$. When the input is $x_2(t) = \delta(t) - e^{-t}U(t)$, the corresponding output is $y_2(t) = \delta(t) + e^{-t}(\cos(t) - \sin(t))U(t)$.

   (a) $\boxed{\text{10 points}}$ What is the impulse response, $h_1(t)$ of system $\mathcal{S}_1$? Do not use the Laplace transform.

   (b) $\boxed{\text{10 points}}$ Suppose now that the impulse response $h_1(t)$ of system $\mathcal{S}_1$ is fed as input to system $\mathcal{S}_2$, whose input-output relation is given by $y(t) = tx(t)$. Compute the impulse response, $h_{1,2}(t,\tau)$ of the system given by the cascade of systems $\mathcal{S}_1$ and $\mathcal{S}_2$. Is the overall system time-invariant? Explain your answer.

TV. $th_1(t-\tau)$

a) $e^{-(t-1)}U(t-1) \longrightarrow \boxed{LTIC} \longrightarrow e^{-(t-1)}\sin(t-1)U(t-1)$

but TI $\Rightarrow$ $\therefore$ $\overset{x(t)}{e^{-t}u(t)} \to \boxed{L,TI,C} \to e^{-t}\sin t\, u(t)$ ✓

$x_2(t) = \delta(t) - e^{-t}u(t) \to \boxed{LTIC} \to \delta(t) + e^{-t}(\cos t - \sin t)\,u(t)$

We know system is linear.

$\therefore\ T_1[x_1(t) + x_2(t)] = T_1[x_1(t)] + T[x_2(t)]$

$T_1[\cancel{e^{-t}u(t)} + \delta(t) - \cancel{e^{-t}u(t)}] = T_1[e^{-t}u(t)] + T[\delta(t) - e^{-t}u(t)]$

$T_1[\delta(t)] = h_1(t) = \cancel{e^{-t}\sin t\, u(t)} + \delta(t) + e^{-t}\cos t\, u(t) - \cancel{e^{-t}\sin t\, u(t)}$

$$\boxed{h_1(t) = \delta(t) + e^{-t}\cos t\, u(t)}$$ ✓

b). $h_1(t) \to \boxed{S_2} \to y(t)$

Time varying since S_2 is TV, and cascade of TI and TV systems is TV. ✓

$\overset{x(t) = \delta(t-\tau)}{\longrightarrow} \boxed{S_1} \to \boxed{S_2} \to$

$h_1(t-\tau) = \delta(t-\tau) + e^{-(t-\tau)}\cos(t-\tau)\,u(t-\tau)$

$y(t) = h_{1,2}(t,\tau) = t h_1(t-\tau)$

$h_{1,2}(t,\tau) = t[\delta(t-\tau) + e^{-(t-\tau)}\cos(t-\tau)u(t-\tau)]$

Time Varying (can't write as function of $t-\tau$).

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$$h_{1,2}(t,\tau) = t\left(\delta(t-\tau) + e^{-(t-\tau)}\cos(t-\tau)\,u(t-\tau)\right)$$

$$= \underbrace{t\,\delta(t-\tau)}_{\text{sifting}} + t\,e^{-(t-\tau)}\cos(t-\tau)\,u(t-\tau)$$

$$\boxed{h_{1,2}(t,\tau) = \tau\,\delta(t-\tau) + t\,e^{-(t-\tau)}\cos(t-\tau)\,u(t-\tau)}$$

time varying (can't write as function of $t-\tau$ $h_{1,2}(t-\tau)$)

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5. $\boxed{\text{10 points}}$ A linear, time-invariant system, has step response given by $g(t) = e^{-t}U(t) - e^{-2t}U(t)$. Determine the output of this system, $y(t)$, given an input $x(t) = \delta(t-\pi) - \cos(\sqrt{3})U(t)$.

$$h(t) = \frac{d}{dt} g(t) = \frac{d}{dt}\left[e^{-t}U(t) - e^{-2t}U(t)\right]$$

$$= -e^{-t}U(t) + e^{-t}\delta(t) - e^{-2t}\delta(t) + 2e^{-2t}U(t)$$

$$= -e^{-t}U(t) + \cancel{\delta(t)} - \cancel{\delta(t)} + 2e^{-2t}U(t)$$

$$h(t) = \boxed{2e^{-2t}U(t) - e^{-t}U(t)} = U(t)\left[2e^{-2t} - e^{-t}\right]$$

$$h(t-\sigma) = U(t-\sigma)\left[2e^{-2(t-\sigma)} - e^{-(t-\sigma)}\right]$$

$$x(\sigma) = \delta(\sigma-\pi) - \cos(\sqrt{3})U(\sigma)$$

$$y(t) = \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)\,d\sigma =$$

$$= \int_{-\infty}^{\infty}\left[2e^{-2(t-\sigma)} - e^{-(t-\sigma)}\right]U(t-\sigma)\left[\delta(\sigma-\pi) - \cos\sqrt{3}\right]U(\sigma)\,d\sigma$$

$$= \int_0^t \left[2e^{-2(t-\sigma)} - e^{-(t-\sigma)}\right]\left[\delta(\sigma-\pi) - \cos\sqrt{3}\right]d\sigma$$

$$= \underbrace{\int_0^t 2e^{-2(t-\sigma)}\delta(\sigma-\pi)\,d\sigma}_{①} - \underbrace{\int_0^t 2e^{-2(t-\sigma)}\cos\sqrt{3}\,d\sigma}_{②} - \underbrace{\int_0^t e^{-(t-\sigma)}\delta(\sigma-\pi)\,d\sigma}_{③} + \underbrace{\int_0^t e^{-(t-\sigma)}\cos\sqrt{3}\,d\sigma}_{④}$$

① $= \int_{-\infty}^{\infty} 2e^{-2(t-\sigma)}\delta(\sigma-\pi)U(t-\sigma)U(\sigma)\,d\sigma = \int_{-\infty}^{\infty} 2e^{-2(t-\pi)}U(t-\pi)\overset{1}{U(\pi)}\delta(\sigma-\pi)\,d\sigma$

$$= 2e^{-2(t-\pi)}U(t-\pi)$$

② $\int_0^t 2e^{-2t}e^{2\sigma}\cos\sqrt{3}\,d\sigma$

$$= 2e^{-2t}\cos\sqrt{3}\int_0^t e^{2\sigma}\,d\sigma$$

$$= e^{-2t}\cos\sqrt{3}\left(e^{2t}-1\right)$$

③ $\int_0^t e^{-(t-\sigma)}\delta(\sigma-\pi)\,d\sigma$

$$= \int_{-\infty}^{\infty} e^{-(t-\sigma)}\delta(\sigma-\pi)U(t-\sigma)U(\sigma)\,d\sigma$$

$$= e^{-(t-\pi)}U(t-\pi)\cancel{U(\pi)}$$

④ $\int_0^t e^{-(t-\sigma)} \cos\sqrt{3}\, d\sigma$

$= \int_0^t e^{-t} e^{\sigma} \cos\sqrt{3}\, d\sigma = e^{-t} \cos\sqrt{3} \int_0^t e^{\sigma} d\sigma$

$= e^{-t} \cos\sqrt{3} (e^{t} - 1)$

$e^{\sigma} \Big|_0^t$

$y(t) = ① - ② - ③ + ④$

$= 2e^{-2(t-\pi)} u(t-\pi) - \underbrace{e^{-2t} \cos\sqrt{3} (e^{2t} - 1)}_{=} - e^{-(t-\pi)} u(t-\pi) + e^{-t} \cos\sqrt{3} (e^{t} - 1)$

$- \cos\sqrt{3} (1 - e^{-2t})$ $\qquad \cos\sqrt{3} (1 - e^{-t})$

$- \cos\sqrt{3} + \cos\sqrt{3} e^{-2t}$ $\qquad \cancel{\cos\sqrt{3}} - \cos\sqrt{3} e^{-t}$

$$\boxed{y(t) = u(t-\pi) \left[2e^{-2(t-\pi)} - e^{-(t-\pi)} \right] + \cos\sqrt{3} \left(e^{-2t} - e^{-t} \right)} u(t)$$

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