**UCLA**
**Dept. of Electrical Engineering**
**EE102: Systems and Signals**
**Midterm Solutions**

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1. *Which of the following systems are linear, time-invariant, causal, and why?*
   (a) $y(t) = x(t+2) - x(t-2)$. Linear: $x(t) \rightarrow k_1 x_1(t) + k_2 x_2(t)$, then output$=$ $k_1(x_1(t+2) - x_1(t-2)) + k_2(x_2(t+2) - x_2(t-2))$. Time-invariant: $x(t) \rightarrow x(t-\tau)$, then output$= x(t+2-\tau) - x(t-2-\tau) = y(t-\tau)$. Non-causal: output at time $t$ depends on the *future*, $x(t+2)$.
   (b) $y(t) = x(t)x(t-1)$. Nonlinear: $x(t) \rightarrow k_1 x_1(t) + k_2 x_2(t)$, then $y \neq k_1 y_1(t) + k_2 y_2(t)$. Time-invariant: $x(t) \rightarrow x(t-\tau)$, then output$= x(t-\tau)x(t-1-\tau) = y(t-\tau)$. Causal: output at time $t$ depends only on input at present and past values of $t$.
   (c) $y(t) = \int_{-\infty}^{t+1} x(\sigma)d\sigma$. Linear: $x(t) \rightarrow k_1 x_1(t) + k_2 x_2(t)$, then $y = k_1 y_1(t) + k_2 y_2(t)$. Time-invariant: $x(t) \rightarrow x(t-\tau)$, then output$= \int_{-\infty}^{t+1} x(\sigma-\tau)d\sigma = \int_{-\infty}^{t-\tau+1} x(u)du =$ $y(t-\tau)$ (upon change of variable $\sigma - \tau = u$). Non-causal: output at time $t$ also depends on values of input in the future, at times larger than $t$.

2. *Find the inverse Laplace transform of the function*

$$F(s) = \frac{e^{-(s-1)} + 3}{s^2 - 2s + 5}, \quad \Re\{s\} > 1.$$

   Start by finding the roots of $s^2 - 2s + 5$, $p_1 = 1 - 2i$, $p_2 = 1 + 2i$: complex conjugate. If we call $X(s) = \frac{1}{s^2 - 2s + 5}$,

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{2}{(s-1)^2 + 4}\right\} = \frac{1}{2}e^t \sin(2t)U(t).$$

   Therefore $F(s) = \frac{e}{2}e^{-s}X(s) + \frac{3}{2}X(s)$ which yields $f(t) = \frac{e}{2}x(t-1) + \frac{3}{2}x(t)$.

3. *The linear, time-invariant, causal system $\mathcal{S}$ admits the output $y(t) = \frac{1}{9}(1 - \cos(3t))U(t)$ when the input is $x(t) = tU(t)$. What is the output $y_1(t)$ when the input is $x_1(t) = U(t)$? Do not use the Laplace transform.*

   The unit step function $x_1(t)$ is simply the derivative of the function $x(t)$, therefore

$$y_1(t) = \mathcal{T}\left\{\frac{d}{dt}x(t)\right\} = \frac{d}{dt}\mathcal{T}\{x(t)\} = \frac{d}{dt}y(t) = \frac{1}{3}\sin(3t)U(t) + \frac{1}{9}(1 - \cos(3t))\delta(t) = \frac{1}{3}\sin(3t)U(t).$$

4. When $x_1(t) = e^{-(t-1)}U(t-1)$ is the input to linear, time-invariant, causal system, $\mathcal{S}_1$, the corresponding output is $y_1(t) = e^{-(t-1)}\sin(t-1)U(t-1)$. When the input is $x_2(t) = \delta(t) - e^{-t}U(t)$, the corresponding output is $y_2(t) = \delta(t) + e^{-t}(\cos(t) - \sin(t))U(t)$.
   (a) *What is the impulse response, $h_1(t)$ of system $\mathcal{S}_1$? Do not use the Laplace transform.*

   Note that $x_2(t) + x_1(t+1) = \delta(t)$, therefore, since the system is linear and time-invariant, $h_1(t) = y_2(t) + y_1(t+1) = \delta(t) + e^{-t}(\cos(t) - \sin(t))U(t) + e^{-t}\sin(t)U(t) =$ $\delta(t) + e^{-t}\cos(t)U(t)$.

(b) *Suppose now that the impulse response $h_1(t)$ of system $\mathcal{S}_1$ is fed as input to system $\mathcal{S}_2$, whose input-output relation is given by $y(t) = tx(t)$. Compute the impulse response, $h_{1,2}(t, \tau)$ of the system given by the cascade of systems $\mathcal{S}_1$ and $\mathcal{S}_2$. Is the overall system time-invariant? Explain your answer.*

The impulse response to the overall system is given by the output of the second system when the input to the *first* system is $\delta(t - \tau)$, i.e., when the input to the *second* system is $h_1(t - \tau)$ (remember that the first system is time-invariant). Therefore

$$h_{1,2}(t, \tau) = th_1(t-\tau) = t(\delta(t-\tau)+e^{-(t-\tau)}\cos(t-\tau)U(t-\tau)) = \tau\delta(t-\tau)+te^{-(t-\tau)}\cos(t-\tau)U(t-\tau).$$

The overall system is time-varying, in fact $h_{1,2}(t, \tau)$ is not solely function of $(t-\tau)$.

5. *A linear, time-invariant system, has step response given by $g(t) = e^{-t}U(t) - e^{-2t}U(t)$. Determine the output of this system, $y(t)$, given an input $x(t) = \delta(t-\pi)-\cos(\sqrt{3})U(t)$.*

   It is known that $h(t) = \frac{d}{dt}g(t) = -e^{-t}U(t) + e^{-t}\delta(t) + 2e^{-2t}U(t) - e^{-2t}\delta(t) = (-e^{-t} + 2e^{-2t})U(t)$. The output $y(t)$ is thus given by

$$y(t) = h(t-\pi)-\cos(\sqrt{3})g(t) = (-e^{-(t-\pi)}+2e^{-2(t-\pi)})U(t-\pi)-\cos(\sqrt{3})(e^{-t}-e^{-2t})U(t).$$