UCLA
Dept. of Electrical Engineering
EE102: Systems and Signals
Midterm Exam
Thursday, May 12, 2005
Please, write your name and ID on the top of each sheet!

① 9
② 10
③ 3
④ 8
⑤ 6
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This exam has 5 questions, for a total of 60 points.

> Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Name and ID: [redacted] [redacted]

Name of person to your left: [redacted]

Name of person to your right: [redacted]

9 1. $\boxed{\text{10 points}}$ Which of the following systems are linear, time-invariant, causal, and why?

(a) $y(t) = x(t+2) - x(t-2)$;

(b) $y(t) = x(t)x(t-1)$;

(c) $y(t) = \int_{-\infty}^{t+1} x(\sigma)d\sigma$.

a) $T[k_1x_1(t) + k_2x_2(t)] \overset{?}{=} k_1T[x_1] + k_2T[x_2]$

$[k_1x_1(t+2) + k_2x_1(t+2)] - [k_1x_1(t-2) + k_2x(t-2)] \overset{?}{=} k_1x_1(t+2) - k_1x_1(t-2) + k_2x_2(t+2) - k_2x_2(t-2)$

$=$ equal so Linear

non-causal because of $x(t+2)$ term (x term from future)

$y(t-6) \overset{?}{=} T[x(t-6)]$

$x(t-6+2) - x(t-6-2) \overset{?}{=} x(t+2-6) - x(t-2-6)$

equal so Time invariant

b) $T[k_1x_1 + k_2x_2] \overset{?}{=} k_1T[x_1] + k_2T[x_2]$

$(k_1x_1(t) + k_2x_2(t))(k_1x_1(t-1) + k_2x_2(t-1)) \overset{?}{=} \neq$

$k_1x_1(t)x_1(t-1) + k_2x_2(t)x_2(t-1)$

not equal so non-linear

Causal b/c $y(t)$ only depends on x values from present + past values of t

$y(t-6) \overset{?}{=} T[x(t-6)]$

$x(t-6)x(t-6-1) \overset{?}{=} x(t-6)x(t-1-6)$

equal so Time invariant

c) $T[k_1x_1 + k_2x_2] \overset{?}{=} k_1T[x_1] + k_2T[x_2]$

$\int_{-\infty}^{t+1} k_1x_1(\sigma) + k_2x_2(\sigma) \overset{?}{=} \int_{-\infty}^{t+1} k_1x_1(\sigma) + \int_{-\infty}^{t+1} k_2x_2(\sigma)$

$=$ equal so linear

non-causal b/c integral takes $x(\sigma) \to x(t+1)$ which is a future value of x

$y(t-6) \overset{?}{=} T[x(t-6)]$

$\int_{-\infty}^{t-6+1} x(\sigma)\, d\sigma = \int_{-\infty}^{t+1} x(\sigma - \tau)\, d\sigma$

$\neq$

not equal so time-varying

10 2. $\boxed{\text{10 points}}$ **Find the inverse Laplace transform of the function**

$s^2-2s+5 \Rightarrow (s-1)^2+4$

$$F(s) = \frac{e^{-(s-1)}+3}{s^2-2s+5}, \quad \Re\{s\} > 1. \tag{1}$$

$= \frac{e^{-(s-1)}}{s^2-2s+5} + \frac{3}{s^2-2s+5} = e^{-(s-1)}\frac{1}{(s-1)^2+4} + \frac{3\,1}{(s-1)^2+4}$ ✓

$\left(\frac{3}{2}\right)\frac{2}{(s-1)^2+2^2} = \left(\frac{3}{2}\right)e^{t}\sin(2t)\,u(t)$

$= e^{-s}e^{1}\left(\frac{1}{2}\right)\frac{2}{(s-1)^2+2^2}$

$= e^{1}e^{-s}\left(\frac{1}{2}\right)\left[e^{t}\sin(2t)\,u(t)\right]$

$= \frac{e}{2}\left[e^{(t-1)}\sin(2t-2)\,u(t-1)\right]$

$\cdot\; e^{-s}X(s) = x(t-1)$

$$\boxed{f(t) = \frac{e^{t}\sin(2t-2)\,u(t-1)}{2} + \frac{3e^{t}\sin(2t)\,u(t)}{2}}$$ ✓

3. $\boxed{10 \text{ points}}$ **The linear, time-invariant, causal system $\mathcal{S}$ admits the output $y(t) = \frac{1}{9}(1 - \cos(3t))U(t)$ when the input is $x(t) = tU(t)$. What is the output $y_1(t)$ when the input is $x_1(t) = U(t)$? Do not use the Laplace transform.**

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$$tu(t) \longrightarrow \boxed{S} \longrightarrow \frac{1}{9}(1-\cos(3t))u(t)$$

$$\frac{dg(t)}{dt} = h(t)$$

$$u(t) \longrightarrow \boxed{S} \longrightarrow y_1(t) = ? = g(t)$$

$$\frac{du(t)}{dt} = \delta(t)$$

$$y(t) = \int_0^t h(t-\sigma)x(\sigma)\,d\sigma \qquad \underline{y(t) = h(t) * x(t)}$$

$$u = \sigma \qquad dv = h(t-\sigma)$$

$$\frac{1}{9}(1-\cos 3t)u(t) = \int_0^t h(t-\sigma)\,\sigma\, u(\sigma)\,d\sigma$$

$$du = d\sigma \qquad v = g(t-\sigma)$$

$$\frac{1}{9}(1-\cos 3t)u(t) = \sigma g(t-\sigma)\Big|_0^t - \int_0^t g(t-\sigma)\,d\sigma$$

$$= tg(0) - 0 - \int_0^t g(t-\sigma)\,d\sigma$$

$$g(t-\sigma)\Big|_0^t \qquad g(0) - g(t)$$

$$\frac{d}{dt}\left(\frac{1}{9}(1-\cos 3t)u(t)\right) = \frac{d}{dt}\left(t - \int_0^t g(t-\sigma)\,d\sigma\right) \quad \text{derive both sides}$$

$$\frac{1}{9}\delta(t) - \frac{1}{9}\left[\cos 3t\,\delta(t) - 3\sin 3t\,u(t)\right] = 1 - g(t)$$

$$\boxed{g(t) = 1 - \left[\frac{1}{9}\delta(t) - \frac{1}{9}\cos 3t\,\delta(t) + \frac{1}{3}\sin(3t)u(t)\right]}$$

4. When $x_1(t) = e^{-(t-1)}U(t-1)$ is the input to linear, time-invariant, causal system, $\mathcal{S}_1$, the corresponding output is $y_1(t) = e^{-(t-1)}\sin(t-1)U(t-1)$. When the input is $x_2(t) = \delta(t) - e^{-t}U(t)$, the corresponding output is $y_2(t) = \delta(t) + e^{-t}(\cos(t) - \sin(t))U(t)$.

(a) $\boxed{\text{10 points}}$ What is the impulse response, $h_1(t)$ of system $\mathcal{S}_1$? Do not use the Laplace transform.

(b) $\boxed{\text{10 points}}$ Suppose now that the impulse response $h_1(t)$ of system $\mathcal{S}_1$ is fed as input to system $\mathcal{S}_2$, whose input-output relation is given by $y(t) = tx(t)$. Compute the impulse response, $h_{1,2}(t,\tau)$ of the system given by the cascade of systems $\mathcal{S}_1$ and $\mathcal{S}_2$. Is the overall system time-invariant? Explain your answer.

$x_1(t) = e^{-(t-1)}u(t-1) \rightarrow \boxed{S_1} \rightarrow e^{-(t-1)}\sin(t-1)u(t-1)$

$x_2(t) = \delta(t) - e^{-t}u(t) \rightarrow \boxed{S_1} \rightarrow \delta(t) + e^{-t}(\cos(t) - \sin(t))u(t)$

LTIC

$h(t) = ?$

a) $y_1(t) = \int_0^t h(t-\sigma)x_1(\sigma)\,d\sigma$

$e^{-(t-1)}\sin(t-1)u(t-1) = \int_0^t h(t)\, e^{-(\sigma-1)}u(\sigma-1)\,d\sigma$

$= e^1 \int_1^t h(t)\, e^{-\sigma}\,d\sigma$

$e^{-t}\sin(t-1)u(t-1) = h(t)\int_1^t e^{-\sigma}\,d\sigma$

$= h(t)(e^{-t} - e^{-1})$

$$\boxed{h_1(t) = \frac{\sin(t-1)u(t-1)e^{-t}}{e^{-t} - e^{-1}}}$$

$y_2(t) = \int_0^t h(t-\sigma)x_2(\sigma)\,d\sigma$

$\delta(t) + e^{-t}(\cos(t) - \sin(t))u(t) = \int_0^t h(t)\,[\delta(\sigma) - e^{-\sigma}u(\sigma)]\,d\sigma$

$= \int_{-\infty}^{\infty} h(t-\sigma)\delta(\sigma) - \int_0^t h(t-\sigma)e^{-\sigma}u(\sigma)\,d\sigma$

$u(\sigma)u(t-\sigma)$

$= h(t)u(t) - h(t)\int_0^t e^{-\sigma}\,d\sigma$

$= h(t)[u(t) + 1 - e^{-t}]$

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b) $y(t) = t\,x(t) = S_2$

$y(t-\sigma) \stackrel{?}{=} T[x(t-\sigma)]$

$(t-\sigma)x(t-\sigma) \neq t\,x(t-\sigma)$

so Not TI

Since S_2 is time varying the overall system is time varying, not TI ✓

If one portion of S is TV, than the whole is TV

$h_2^{(t)} = t\delta(t)$

$h_2(t,\sigma) = t\delta(t-\sigma)$

$h_{12}^{(t,\tau)} = \int_{-\infty}^{\infty} h_2(t,\sigma)\,h_1(\sigma)\,d\sigma$

$h_{12} = \int_{-\infty}^{\infty} t\delta(t-\sigma)\, e^{-t}\sin(\sigma-1)u(\sigma-1)/(e^{-t} - e^{-1})\,d\sigma$

sifting

$$\boxed{h_{1,2}^{(t,\tau)} = \frac{t\sin(t-1)u(t-1)e^{-t}}{e^{-t} - e^{-1}}}$$

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5. $\boxed{\text{10 points}}$ A linear, time-invariant system, has step response given by $g(t) = e^{-t}U(t) - e^{-2t}U(t)$. Determine the output of this system, $y(t)$, given an input $x(t) = \delta(t-\pi) - \cos(\sqrt{3})U(t)$.

$g(t) = e^{-t}u(t) - e^{-2t}u(t)$

$\frac{d}{dt}g(t) = h(t) = -e^{-t}u(t) + e^{-t}\delta(t) - [-2e^{-2t}u(t) + e^{-2t}\delta(t)]$ ✓

$= \cancel{e^{-t}\delta(t)} + 2e^{-2t}u(t) - e^{-t}u(t) - \cancel{e^{-2t}\delta(t)}$ ✓

$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)\,d\sigma$ ✓

$Y(s) = H(s)X(s)$ ✓

$H(s) = 2e^{-2t}u(t) - e^{-t}u(t)$

$= 2\frac{1}{s+2} - \frac{1}{s+1}$ ✓

$X(s) = \delta(t-\pi) - \cos(\sqrt{3})\,u(t)$

$= 1e^{-s\pi} - \frac{s}{s^2+3}$ ✗

$\cos\sqrt{3} \neq \cos(\sqrt{3}t)$

$H(s)X(s) = \left(\frac{2}{s+2} - \frac{1}{s+1}\right)\left(e^{-s\pi} - \frac{s}{s^2+3}\right)$

$= \frac{2e^{-s\pi}}{s+2} - \frac{e^{-s\pi}}{s+1} - \frac{2s}{(s^2+3)(s+2)} + \frac{s}{(s^2+3)(s+1)}$

(A) $= 2e^{-s\pi}\frac{1}{s+2}$

$= e^{-2t}$

$= 2e^{-2(t-\pi)}$

$u(t-\pi)$

(B) $= -e^{-s\pi}\frac{1}{s+1}$

$= -e^{-t}$

$= -e^{-(t-\pi)}$

$u(t-\pi)$

(C) $\frac{2s}{(s^2+3)(s+2)} = \frac{As+B}{s^2+3} + \frac{C}{s+2}$

$4/7 \quad 6/7 \quad -4/7$

$\left.\frac{2s}{s^2+3}\right|_{-2} = C = \frac{-4}{4+3} = -4/7$

$\frac{2s^2}{\sim} = \frac{As^2+Bs}{s^2+3} + \frac{Cs}{s+2} \Rightarrow \lim_{s\to\infty} sY(s)$

$0 = A + C, \quad A = -C = 4/7$

$0 = \frac{B}{3} + \frac{C}{2} \quad |_{s=0}$

$-3C = 2B$

$B = -\frac{3}{2}(-4/7) = 6/7$

$\sqrt{3} \quad \frac{\sqrt{3}\cdot 2}{7}$

$= \frac{4/7\,s + //}{s^2+3} - \frac{4/7}{s+2} + \frac{6/7}{s^2+3}$

$= \frac{4}{7}\cos(\sqrt{3}t)u(t) - \frac{4}{7}e^{-2t}u(t) + \frac{2\sqrt{3}}{7}\sin(\sqrt{3}t)u(t)$

D

$\sqrt{3} \cdot \frac{\sqrt{3}}{4}$

$$\frac{s}{(s^2+3)(s+1)} = \frac{As+B}{s^2+3} + \frac{C}{s+1} = \frac{1/4\, s}{s^2+3} + \frac{3/4}{s^2+3} - \frac{1/4}{s+1}$$

$$= \frac{1}{4}\cos(\sqrt{3}t)u(t) - 1/4\, e^{-t}u(t) + \frac{\sqrt{3}}{4}\sin(\sqrt{3}t)u(t)$$

$$\left.\frac{s}{s^2+3}\right|_{-1} = C = -\frac{1}{4}$$

$$\lim_{s\to\infty} sY(s) = 0 = A + C \qquad A = 1/4$$

$$s=0 \;\Big|\; 0 = \frac{B}{3} + C$$

$$\frac{B}{3} = 1/4 \qquad B = 3/4$$

$$\frac{1}{4} - \frac{4}{7} = \frac{7}{28} - \frac{16}{28} = \frac{-9}{28} = -\frac{9}{28}$$

$$\frac{\sqrt{3}}{4} - \frac{2\sqrt{3}}{7} = \frac{7\sqrt{3}}{28} - \frac{8\sqrt{3}}{28} = -\frac{\sqrt{3}}{28}$$

$$\boxed{\begin{aligned} &y(t) \\ &\text{Sum} = 2e^{-7(t-\pi)}u(t-\pi) - e^{-(t-\pi)}u(t-\pi) + \frac{4}{7}e^{-2t}u(t) - 1/4\, e^{-t}u(t) - \frac{9}{28}\cos(\sqrt{3}t)u(t) \\ &\qquad - \frac{\sqrt{3}}{28}\sin(\sqrt{3}t)u(t) \end{aligned}}$$