UCLA — Electrical Engineering Dept.
EE102: Systems and Signals — Midterm Exam
Wednesday, October 28, 2015

This exam has 4 questions, for a total of 33 points.

Closed book. One two-sided cheat-sheet allowed.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
Please, write your name and ID on the top of each loose sheet!

Name and ID: ______________________________

Name of person on your left: ______________________________

Name of person on your right: ______________________________

Question	Points	Score
1	9	7
2	10	7
3	8	8
4	6	6
Total:	33	28

Multiple-choice questions – Check all the answers that apply.

1. (a) (3 points) Consider the system described by the input-output relation

$$y(t) = \sin(x(t+1)).$$

What are its properties?

- [ ] Linear
- [x] Time-invariant
- [ ] Causal

✓

(b) (3 points) Consider the system described by the input-output relation

$$y(t) = \int_{-\infty}^{\infty} e^{t} e^{-\tau} x(\tau) u(t-\tau)\, d\tau.$$

What are its properties?

$$e^{t} \int_{-\infty}^{\infty} e^{-\tau} x(\tau) u(t-\tau)\, d\tau$$

- [x] Linear
- [ ] Time-invariant
- [x] Causal

-1

(c) (3 points) Consider the system described by the input-output relation

$$y(t) = x(at), \quad a > 0.$$

~~What is the~~ condition on a for the system to be causal?

- [x] $a < 1$
- [x] $a = 1$
- [ ] $a > 1$

-1

2. The input-output relation of a system $\mathcal{S}$ is

$$y(t) = \int_{-\infty}^{\infty} \sin(t-\tau)\mathrm{u}(t-2\tau)x(\tau)\,\mathrm{d}\tau.$$

(a) (2 points) Is the system time-invariant or time-varying?

(b) (3 points) What is the impulse response of the system?

(c) (5 points) Compute the output, $y(t)$, when the input is $x(t) = \mathrm{u}(t) - \mathrm{u}(t-1)$

(a) Time Varying. It depends on $(t-2\tau)$, not only $(t-\tau)$.

what depends on $t-2\tau$?

$\boxed{2}$

(b) $h(t;\sigma) = \int_{-\infty}^{\infty} \sin(t-\tau)u(t-2\tau)\delta(\tau-\sigma)\,d\tau$

$x(t)=h(t-\sigma)$

$x(\tau)=h(\tau-\sigma)$

$= \int_{-\infty}^{\frac{1}{2}t} \sin(t-\tau)\delta(\tau-\sigma)\,d\tau$

if $t>2\tau$, τ then:

if $\sigma<\frac{1}{2}t$, then

$h(t;\sigma) = \sin(t-\sigma)$

ie. $h(t;\sigma) = \sin(t-\sigma)$ ok

So: $h(t;\sigma) = \sin(t-\sigma)\,u(t-2\sigma)$

$\boxed{3}$

(c) Because $T[k_1x_1(t)+k_2x_2(t)] = k_1T[x_1(t)] + k_2T[x_2(t)]$

So the system is linea

$y(t) = T[u(t)-u(t-1)] = T[u(t)] - T[u(t-1)]$

$\frac{d}{dt}y(t) = T[\delta(t)] + T[\delta(t-1)] = h(t;0) - h(t;1)$

$\frac{d}{dt}y(t) = \sin(t)u(t) - \sin(t-1)u(t-2)$

$y(t) = \int_{-\infty}^{t} \sin(\tau)u(\tau)\,d\tau - \int_{-\infty}^{t} \sin(\tau-1)u(\tau-2)\,d\tau$

$$y(t) = \int_{-\infty}^{t} \sin(\tau)u(\tau)\,d\tau - \int_{-\infty}^{t} \sin(\tau-1)\,u(\tau-2)\,d\tau$$

$$= \int_{0}^{t} \sin(\tau)\,d\tau - \int_{2}^{t} \sin(\tau-1)\,d\tau$$

$$= -\cos(\tau)\Big|_{0}^{t} + \cos(\tau-1)\Big|_{2}^{t}$$

$$= -\cos(t) + 1 + \cos(t-1) - \cos(1)$$

$$= \cos(t-1) - \cos(t) + 1 - \cos(1)$$

↓ incorrect, need not have ~~used~~ ~~[illegible]~~. done this all.

$\boxed{2.}$

~~1) Systems not LTI~~

3. Let $\mathcal{S}$ be a linear, time-invariant, and causal system. We know that the output corresponding ✓ to $x(t) = (t-3)\mathrm{u}(t-3)$ is

$$y(t) = 4\mathrm{e}^{-(t-3)}\mathrm{u}(t-3) - (t-4)\mathrm{u}(t-4).$$

(a) (3 points) What is the impulse response of $\mathcal{S}$? (Hint: consider that $\frac{\mathrm{d}}{\mathrm{d}t}t\mathrm{u}(t) = \mathrm{u}(t)$.)

(b) (5 points) Compute the output of $\mathcal{S}$ when $x(t) = (1-t)\mathrm{u}(t)$.

(a) $y(t) = T[x(t)]$

$y(t+3) = T[x(t+3)] \to$ Time-invariant

$4e^{-t}u(t) - (t-1)u(t-1) = T[tu(t)]$

$\frac{d}{dt}T[tu(t)] = T[\frac{d}{dt}tu(t)] = T[u(t)]$

$T[u(t)] = 4e^{-t}\delta(t) - 4e^{-t}u(t) - \cancel{(t-1)\delta(t-1)}^{\to 0} - u(t-1)$

$= 4\delta(t) - 4e^{-t}u(t) - u(t-1)$ ✓

$\frac{d}{dt}T[u(t)] = T[\frac{d}{dt}u(t)] = T[\delta(t)] = h(t)$

$h(t) = 4\delta'(t) - 4e^{-t}\delta(t) + 4e^{-t}u(t) - \delta(t-1)$

(2) $y_1(t) = T[(1-t)u(t)] = T[u(t) - tu(t)] = T[u(t)] - T[tu(t)]$

$y_1(t) = 4\delta(t) - 4e^{-t}u(t) - u(t-1) - 4e^{-t}u(t) + (t-1)u(t-1)$

$= 4\delta(t) - 8e^{-t}u(t) + (t-2)u(t-1)$

✓

4. A system $\mathcal{S}_1$ is described by

$$\frac{dy}{dt} + 2y(t) = \frac{dx}{dt} - 2x(t), \quad t > 0, x(0) = 0, y(0) = 0.$$

(a) (3 points) Write down the impulse response function $h(t;\tau)$ of $\mathcal{S}_1$.

(b) (3 points) System $\mathcal{S}_1$ is now cascaded with a second linear, time-invariant system, $\mathcal{S}_2$ whose unit step response, $g_2(t)$ is given by LTI S_2

$$g_2(t) = te^{-t}u(t). = T[u(t)]$$

Compute the impulse response, $h_{1,2}(t;\tau)$, of the cascaded combination.

(a)
$$e^{2t}\frac{dy}{dt} + 2e^{2t}y(t) = e^{2t}\frac{dx}{dt} + 2e^{2t}x(t) - 4e^{2t}x(t)$$

$$\frac{d}{dt}[e^{2t}y(t)] = \frac{d}{dt}[e^{2t}x(t)] - 4e^{2t}x(t)$$

$$e^{2t}y(t) = e^{2t}x(t) - \int_{-\infty}^{t} 4e^{2\sigma}x(\sigma)\,d\sigma$$

$$y(t) = x(t) - e^{-2t}\int_{-\infty}^{t} 4e^{2\sigma}x(\sigma)\,d\sigma \checkmark$$

$$h_1(t;\tau) = \delta(t-\tau) - e^{-2t}\int_{-\infty}^{\infty} e^{2\sigma}\delta(\sigma-\tau)\,d\sigma \quad u(t-\sigma)$$

$$= \delta(t-\tau) - e^{-2t}e^{2\tau}u(t-\tau) = \delta(t-\tau) - e^{-2(t-\tau)}u(t-\tau) \checkmark$$

(b) $h_2(t) = T_2[\delta(t)] = \frac{d}{dt}T_2[u(t)] = \frac{d}{dt}g_2(t).$

$$= e^{-t}u(t) + t[-e^{-t}u(t) + e^{-t}\delta(t)] = e^{-t}u(t) - te^{-t}u(t) = (1-t)e^{-t}u(t) \checkmark$$

$$h_{1,2}(t;\tau) = T_2[h_1(t;\tau)] = T_2[\delta(t-\tau)] - T_2[e^{-2(t-\tau)}u(t-\tau)] = h_2(t-\tau) - T_2[e^{-2(t-\tau)}u(t-\tau)]$$

$$T_2[e^{-2(t-\tau)}u(t-\tau)] = \int_{-\infty}^{\infty} e^{-2(t-\tau)}u(t-\tau)\,(1-(t-\tau))e^{-(t-\tau)}u(t-\tau)\,d\tau$$

$$= e^{-3t}\left[\int_{-\infty}^{t} e^{3\tau}\,d\tau - t\int_{-\infty}^{t} e^{3\tau}\,d\tau + \int_{-\infty}^{t} e^{3\tau}\tau\,d\tau\right]$$

$$= e^{-3t}\left[\frac{1}{3}e^{3t} - \cancel{\tfrac{1}{3}te^{3t}} + \cancel{\tfrac{1}{3}te^{3t}} - \frac{1}{9}e^{3t}\right] \Rightarrow \text{Next Page}$$

Please go to next page...

$$\cancel{h_{1,2}(t;\tau) = (1-0)e^{-0}u(0) \int_{-\infty}^{\infty} e^{-2(t-\tau)} [(1-(t-\tau))e^{-(t-\tau)} u(t-}$$

$$\cancel{= 1 - \int_{-\infty}^{\infty} e^{-3(t-\tau)} u(t-\tau)\,d\tau + \int_{-\infty}^{\infty} e^{-3(t-\tau)} u(t-\tau)(t-\tau)\,d}$$

$$h_{1,2}(t;\tau) = (1-(t-\tau))\,e^{-(t-\tau)}u(t-\tau) - \frac{1}{3} + \frac{1}{9}$$

$$= (1-(t-\tau))\,e^{-(t-\tau)}u(t-\tau) - \frac{2}{9}$$ OK

Product to Sum

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$
$$2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta)$$
$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$
$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

Sum to Product

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\tfrac{\alpha+\beta}{2}\right)\cos\left(\tfrac{\alpha-\beta}{2}\right)$$
$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\tfrac{\alpha+\beta}{2}\right)\sin\left(\tfrac{\alpha-\beta}{2}\right)$$
$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\tfrac{\alpha+\beta}{2}\right)\cos\left(\tfrac{\alpha-\beta}{2}\right)$$
$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\tfrac{\alpha+\beta}{2}\right)\sin\left(\tfrac{\alpha-\beta}{2}\right)$$

Sum/Difference

$$\sin(\alpha\pm\beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha\pm\beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$
$$\tan(\alpha\pm\beta) = \frac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$$

Pythagorean Identity

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$

Even/Odd

$$\sin(-\alpha) = -\sin(\alpha)$$
$$\cos(-\alpha) = \cos(\alpha)$$
$$\tan(-\alpha) = -\tan(\alpha)$$

Periodic Identities

$$\sin(\alpha+2\pi n) = \sin(\alpha)$$
$$\cos(\alpha+2\pi n) = \cos(\alpha)$$
$$\tan(\alpha+\pi n) = \tan(\alpha)$$

Double-Angle Identities

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$
$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$
$$\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$$

Half-Angle Identities

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}$$
$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}}$$
$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}}$$

Laws of Sines, Cosines, and Tangents

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$
$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$
$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$$

Mollweide's Formula

$$\frac{a+b}{c} = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\gamma}{2}\right)}$$