Solutions

Multiple-choice questions – Check all the answers that apply.

1. (a) (3 points) Consider the system described by the input-output relation

$$y(t) = \sin(x(t+1)).$$

What are its properties?

☐ Linear
✓ **Time-invariant**
☐ Causal

(b) (3 points) Consider the system described by the input-output relation

$$y(t) = \int_{-\infty}^{\infty} \mathrm{e}^{t}\mathrm{e}^{-\tau} x(\tau)\mathrm{u}(t-\tau)\,\mathrm{d}\tau.$$

What are its properties?

✓ **Linear**
✓ **Time-invariant**
✓ **Causal**

(c) (3 points) Consider the system described by the input-output relation

$$y(t) = x(at), \quad a > 0.$$

What is the condition on a for the system to be causal?

☐ $a < 1$
✓ $a = 1$
☐ $a > 1$

2. The input-output relation of a system $\mathcal{S}$ is

$$y(t) = \int_{-\infty}^{\infty} \sin(t-\tau)\mathrm{u}(t-2\tau)x(\tau)\,\mathrm{d}\tau.$$

(a) (2 points) Is the system time-invariant or time-varying?

Solution: Time-varying:

$$\begin{aligned}\int_{-\infty}^{\infty} \sin(t-\tau)\mathrm{u}(t-2\tau)x(\tau-\sigma)\,\mathrm{d}\tau &= \int_{-\infty}^{\infty} \sin(t-\sigma-\theta)\mathrm{u}(t-2\sigma-2\theta)x(\theta)\,\mathrm{d}\theta \\ &\neq \int_{-\infty}^{\infty} \sin(t-\sigma-\tau)\mathrm{u}(t-\sigma-2\tau)x(\tau)\,\mathrm{d}\tau.\end{aligned}$$

(b) (3 points) What is the impulse response of the system?

Solution: Set $x(t) = \delta(t-\sigma)$.

$$h(t;\sigma) = \sin(t-\sigma)\mathrm{u}(t-2\sigma).$$

(c) (5 points) Compute the output, $y(t)$, when the input is $x(t) = \mathrm{u}(t) - \mathrm{u}(t-1)$

Solution:
For $t < 0, y(t) = 0$ since

$$y(t) = \int_0^1 \sin(t-\tau)\mathrm{u}(t-2\tau)\,\mathrm{d}\tau = 0.$$

For $0 \le t < 2$:

$$\begin{aligned}y(t) &= \int_0^1 \sin(t-\tau)\mathrm{u}(t-2\tau)\,\mathrm{d}\tau \\ &= \int_0^{t/2} \sin(t-\tau)\,\mathrm{d}\tau \\ &= \int_{t/2}^{t} \sin\theta\,\mathrm{d}\theta \\ &= \cos(t/2) - \cos(t).\end{aligned}$$

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For $t \geq 2$:

$$\begin{aligned} y(t) &= \int_0^1 \sin(t-\tau)\mathrm{u}(t-2\tau)\,\mathrm{d}\tau \\ &= \int_0^1 \sin(t-\tau)\,\mathrm{d}\tau \\ &= \int_{t-1}^{t} \sin\theta\,\mathrm{d}\theta \\ &= \cos(t-1)-\cos(t). \end{aligned}$$

3. Let $\mathcal{S}$ be a linear, time-invariant, and causal system. We know that the output corresponding to $x(t) = (t-3)\mathrm{u}(t-3)$ is

$$y(t) = 4\mathrm{e}^{-(t-3)}\mathrm{u}(t-3) - (t-4)\mathrm{u}(t-4).$$

(a) (3 points) What is the impulse response of $\mathcal{S}$? (Hint: consider that $\frac{\mathrm{d}}{\mathrm{d}t} t\mathrm{u}(t) = \mathrm{u}(t)$.)

Solution: Shift the output by 3 to yield:

$$T\left[t\,\mathrm{u}(t)\right] = 4\mathrm{e}^{-t}\mathrm{u}(t) - (t-1)\mathrm{u}(t-1).$$

Taking derivatives we get

$$\begin{aligned} T\left[\mathrm{u}(t)\right] &= -4\mathrm{e}^{-t}\mathrm{u}(t) + 4\mathrm{e}^{-t}\delta(t) - \mathrm{u}(t-1) - (t-1)\delta(t-1) \\ &= -4\mathrm{e}^{-t}\mathrm{u}(t) + 4\delta(t) - \mathrm{u}(t-1) =: g(t), \\ T\left[\delta(t)\right] &= 4\mathrm{e}^{-t}\mathrm{u}(t) - 4\mathrm{e}^{-t}\delta(t) + 4\delta'(t) - \delta(t-1) \\ &= 4\mathrm{e}^{-t}\mathrm{u}(t) - 4\delta(t) + 4\delta'(t) - \delta(t-1) =: h(t). \end{aligned}$$

(b) (5 points) Compute the output of $\mathcal{S}$ when $x(t) = (1-t)\mathrm{u}(t)$.

Solution:

$$\begin{aligned} T\left[(1-t)\mathrm{u}(t)\right] &= T\left[\mathrm{u}(t)\right] - T\left[t\,\mathrm{u}(t)\right] \\ &= -4\mathrm{e}^{-t}\mathrm{u}(t) + 4\delta(t) - \mathrm{u}(t-1) - 4\mathrm{e}^{-t}\mathrm{u}(t) + (t-1)\mathrm{u}(t-1) \\ &= -8\mathrm{e}^{-t}\mathrm{u}(t) + 4\delta(t) + (t-2)\mathrm{u}(t-1). \end{aligned}$$

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4. A system $\mathcal{S}_1$ is described by

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2y(t) = \frac{\mathrm{d}x}{\mathrm{d}t} - 2x(t), \quad t > 0, x(0) = 0, y(0) = 0.$$

(a) (3 points) Write down the impulse response function $h_1(t;\tau)$ of $\mathcal{S}_1$.

Solution:

$$\begin{aligned}
\mathrm{e}^{-2t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2t}y(t)\right) &= \mathrm{e}^{2t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-2t}x(t)\right) \quad \text{(two integrating factors)} \\
\mathrm{e}^{2t}y(t) &= \int_0^t \mathrm{e}^{4\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathrm{e}^{-2\tau}x(\tau)\right)\mathrm{d}\tau \\
&= \mathrm{e}^{2t}x(t) - 4\int_0^t \mathrm{e}^{2\tau}x(\tau)\,\mathrm{d}\tau \\
\therefore y(t) &= x(t) - 4\int_0^t \mathrm{e}^{-2(t-\tau)}x(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

Setting $x(t) = \delta(t-\sigma)$

$$h_1(t;\sigma) = \delta(t-\sigma) - 4\mathrm{e}^{-2(t-\sigma)}\mathrm{u}(t-\sigma), \quad t > 0.$$

The system is time-invariant, $t > 0$, and $h_1(t) = \delta(t) - 4\mathrm{e}^{-2t}\mathrm{u}(t)$.

(b) (3 points) System $\mathcal{S}_1$ is now cascaded with a second linear, time-invariant system, $\mathcal{S}_2$ whose unit step response, $g_2(t)$ is given by

$$g_2(t) = t\,\mathrm{e}^{-t}\mathrm{u}(t).$$

Compute the impulse response, $h_{1,2}(t;\tau)$, of the cascaded combination.

Solution: Upon derivation,

$$h_2(t) = \mathrm{e}^{-t}\mathrm{u}(t) - t\,\mathrm{e}^{-t}\mathrm{u}(t) + t\,\mathrm{e}^{-t}\delta(t) = (1-t)\,\mathrm{e}^{-t}\mathrm{u}(t).$$

For $t > 0$ both systems are time-invariant, so, for $t > 0$:

$$\begin{aligned}
h_{1,2}(t) &= (h_1 * h_2)(t) \\
&= (1-t)\,\mathrm{e}^{-t} - 4\int_0^t (1-\tau)\mathrm{e}^{-\tau}\mathrm{e}^{-2(t-\tau)}\,\mathrm{d}\tau \\
&= (1-t)\,\mathrm{e}^{-t} - 4\mathrm{e}^{-2t}\int_0^t (1-\tau)\mathrm{e}^{\tau}\,\mathrm{d}\tau \\
&= (1-t)\,\mathrm{e}^{-t} - 4\mathrm{e}^{-2t}\left((1-\tau)\mathrm{e}^{\tau}\Big|_0^t + \int_0^t \mathrm{e}^{\tau}\,\mathrm{d}\tau\right) \quad \text{(IBP)} \\
&= (1-t)\,\mathrm{e}^{-t} - 4\mathrm{e}^{-2t}\left((1-t)\mathrm{e}^{t} - 1 + \mathrm{e}^{t} - 1\right) \\
&= -3(1-t)\,\mathrm{e}^{-t} + 8\mathrm{e}^{-2t} - 4\mathrm{e}^{-t}.
\end{aligned}$$