Solutions

1. A linear system $\mathcal{S}_1$ is described by the following system equation

$$H_1(s) = \frac{s-1}{s+2}, \quad \text{ROC} : \text{Re}(s) > -2.$$

(a) (10 points) Write down the differential equation relating input, $x(t)$, and output, $y(t)$, of the system $\mathcal{S}_1$.

(b) (10 points) Without using the Laplace transform, find the impulse response function $h_{1_\tau}(t)$ of $\mathcal{S}_1$. (Assume all-zero initial conditions!)

(c) (10 points) System $\mathcal{S}_1$ is now cascaded with a second linear, time-invariant system $\mathcal{S}_2$, whose unit step response is given by

$$g_2(t) = \frac{1}{2}t^2\text{u}(t).$$

Compute the impulse response of $\mathcal{S}_2$.

(d) (10 points) Compute the impulse response, $h_{1,2_\tau}(t)$, of the cascaded combination.

Solution:

$$H_1(s) = 1 + \frac{-3}{s+2}.$$

(a)

$$\frac{Y(s)}{X(s)} = H_1(s) \quad y'(t) + 2y(t) = x'(t) - x(t).$$

(b) LTIC, therefore $h_{1_\tau}(t) = h_1(t-\tau)$.

$$\frac{\mathrm{d}y(t)\mathrm{e}^{2t}}{\mathrm{d}t} = \mathrm{e}^{3t}\frac{\mathrm{d}x(t)\mathrm{e}^{-t}}{\mathrm{d}t}.$$

Upon integration:

$$y(t)\mathrm{e}^{2t} = \mathrm{e}^{2t}x(t) - 3\int_0^t x(\tau)\mathrm{e}^{2\tau}\,\mathrm{d}\tau,$$

hence

$$h_1(t) = \delta(t) - 3\mathrm{e}^{-2t}\text{u}(t).$$

(c)

$$g_2(t) = \frac{1}{2}t^2\text{u}(t) \quad h_2(t) = \frac{1}{2}2t\text{u}(t) + \frac{1}{2}t^2\delta(t) = t\text{u}(t).$$

(d) Because $h_{1,2}(t) = h_1(t) \star h_2(t)$, and $H_1(s) = 1 - \frac{3}{s+2}$, $H_2(s) = \frac{1}{s^2}$,

$$H_{1,2}(s) = \frac{s-1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2},$$

with

$$B = \left.\frac{s-1}{s+2}\right|_{s=0} = -\frac{1}{2}, \quad C = \left.\frac{s-1}{s^2}\right|_{s=-2} = -\frac{3}{4},$$

A can be found by equating the two expressions for $H_{1,2}(s)$, which yields $A = \frac{3}{4}$. Therefore

$$h_{1,2}(t) = \left(\frac{3}{4} - \frac{1}{2}t - \frac{3}{4}\mathrm{e}^{-2t}\right)\mathrm{u}(t).$$

2. The input, $x(t)$, and output, $y(t)$, of a linear, time-invariant, causal system $\mathcal{S}$ are related by the following differential equation

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 4y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}, \quad x(t) = y(t) = x'(t) = y'(t) = 0, t \leq 0.$$

(a) (10 points) Write down the system function, $H(s)$, of $\mathcal{S}$.

(b) (10 points) Compute the impulse response function, $h(t)$ of $\mathcal{S}$.

(c) (10 points) Compute the output, $y(t)$, corresponding to the input $x(t) = \mathrm{e}^{-t}\mathrm{u}(t)$.

Solution:

$$(s^2 + 4s + 4)Y(s) = sX(s).$$

(a)

$$H(s) = \frac{s}{s^2 + 4s + 4} = \frac{2}{(s+2)^2}.$$

(b)

$$H(s) = \frac{A}{(s+2)^2} + \frac{B}{s+2}.$$

$A = s|_{s=-2} = -2$, therefore $B = 1$. Hence

$$H(s) = \frac{-2}{(s+2)^2} + \frac{1}{s+2} \quad \Rightarrow \quad h(t) = -2t\mathrm{e}^{-2t}\mathrm{u}(t) + \mathrm{e}^{-2t}\mathrm{u}(t).$$

(c) $X(s) = \frac{1}{s+1}$ and $Y(s) = H(s)X(s) = \frac{s}{(s+1)(s+2)^2}$ and

$$Y(s) = \frac{A}{(s+2)^2} + \frac{B}{s+2} + \frac{C}{s+1}.$$

We have $C = -1$, $A = 2$, and finally $B = 1$, which yield

$$y(t) = \left(2t\mathrm{e}^{-2t} + \mathrm{e}^{-2t} - \mathrm{e}^{-t}\right)\mathrm{u}(t).$$

3. A linear, time-invariant, causal system $\mathcal{S}$ has impulse response

$$h(t) = \delta(t) + te^{-t}u(t).$$

(a) (10 points) Compute the system function, $H(s)$, of $\mathcal{S}$.

(b) (10 points) Compute the output, $y(t)$, corresponding to the input $x(t) = e^{-t}u(t-1)$.

Solution: $h(t) = \delta(t) + te^{-t}u(t)$.

(a)

$$H(s) = 1 + \frac{1}{(s+1)^2} = \frac{s^2+2s+3}{(s+1)^2}.$$

(b) Let $x_0(t) = e^{-t}u(t)$. Note that $x(t) = e^{-1}x_0(t-1)$. The corresponding output $y_0(t)$ is obtained by convolution:

$$y_0(t) = \int_{-\infty}^{\infty} \left[\delta(t-\tau) + (t-\tau)e^{-(t-\tau)}u(t-\tau)\right] e^{-\tau}u(\tau)\, d\tau = e^{-t}\left(1+\frac{t^2}{2}\right)u(t).$$

$$y(t) = e^{-1}y_0(t-1) = e^{-t}\left(1+\tfrac{(t-1)^2}{2}\right)u(t-1).$$

4. (10 points) Compute the Laplace transform of

$$z(t) = \int_0^t \mathrm{e}^{-2\sigma} y(\sigma)\ \mathrm{d}\sigma,$$

where $y(t)$ is the output of a linear, time-invariant system with impulse response $h(t) = \mathrm{e}^{-t}\cos(t)\mathrm{u}(t)$, when the input is given by $x(t) = \sin(2t)\mathrm{u}(t)$.

Solution:

$$z(t) = \int_0^t \mathrm{e}^{-2\sigma} y(\sigma)\, \mathrm{d}\sigma \quad \Rightarrow \quad Z(s) = \frac{1}{s} Y(s+2).$$

$H(s) = \frac{s+1}{(s+1)^2+1}$, $X(s) = \frac{2}{s^2+4}$, $Y(s) = \frac{2(s+1)}{[(s+1)^2+1](s^2+4)}$, and

$$Z(s) = \frac{2(s+3)}{s[(s+3)^2+1][(s+2)^2+4]}.$$

5. A time-invariant system $\mathcal{S}$ is described by the input-output relation

$$y(t) = \int_0^1 \mathrm{e}^{-2\sigma} x(t-\sigma)\, \mathrm{d}\sigma.$$

(a) (10 points) Is the system linear? Is it causal? Give **brief** explanations to your answers.

(b) (10 points) Write down the impulse response function, $h(t)$.

(c) (10 points) Write down the system function, $H(s)$.

(d) (10 points) Write down the unit step response function, $g(t)$.

Solution:

(a) Linear (integral); causal ($y(t)$ is obtained from values of $x(\sigma)$ for $\sigma < t$).

(b)

$$y(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-2\sigma} x(t-\sigma)\, [\mathrm{u}(\sigma) - \mathrm{u}(\sigma-1)]\, \mathrm{d}\sigma,$$

$h(t) = \mathrm{e}^{-2t}\, [\mathrm{u}(t) - \mathrm{u}(t-1)]$.

(c)

$$H(s) = \int_0^1 \mathrm{e}^{-2t} \mathrm{e}^{-st}\, \mathrm{d}t = \frac{1 - \mathrm{e}^{-(s+2)}}{s+2}.$$

(d)

$$g(t) = \int_0^t \mathrm{e}^{-2\sigma}\, [\mathrm{u}(\sigma) - \mathrm{u}(\sigma-1)]\, \mathrm{d}\sigma, \quad t \geq 0,$$

which yields

$$g(t) = \begin{cases} \frac{1-\mathrm{e}^{-2t}}{2}, & 0 < t < 1, \\ \frac{1-\mathrm{e}^{-2}}{2}, & t > 1, \\ 0, & t < 0. \end{cases}$$