Solutions

1. Consider the system described by the following differential equation:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} - \frac{\mathrm{e}^{-t}}{1+\mathrm{e}^{-t}}\,y(t) = x(t), \quad t > 0, \quad y(0) = 0$$

(a) (5 points) Find $h_\tau(t)$, the impulse response of the system above.

(b) (5 points) Is it a linear system?

(c) (5 points) Is it a time-invariant system?

(d) (5 points) Is it a causal system?

Solution:

(a)

$$(1+\mathrm{e}^{-t})\frac{\mathrm{d}y(t)}{\mathrm{d}t} - \mathrm{e}^{-t}y(t) = (1+\mathrm{e}^{-t})x(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[(1+\mathrm{e}^{-t})y(t)] = (1+\mathrm{e}^{-t})x(t)$$

$$(1+\mathrm{e}^{-t})y(t) = \int_0^t (1+\mathrm{e}^{-\tau})x(\tau)\,\mathrm{d}\tau.$$

$$y(t) = \frac{1}{(1+\mathrm{e}^{-t})}\int_{-\infty}^{\infty}(1+\mathrm{e}^{-\tau})\mathrm{u}(\tau)\mathrm{u}(t-\tau)x(\tau)\,\mathrm{d}\tau.$$

$$\therefore h_\tau(t) = \frac{1+\mathrm{e}^{-\tau}}{1+\mathrm{e}^{-t}}\mathrm{u}(\tau)\mathrm{u}(t-\tau)$$

(b) It is obvious that

$$\mathrm{T}[\alpha x_1(t) + \beta x_2(t)] = \alpha\mathrm{T}[x_1(t)] + \beta\mathrm{T}[x_2(t)]$$

Therefore, the system is linear.

(c) Since $h_\tau(t)$ is not a function of $(t-\tau)$, the system is NOT time-invariant.

(d) From the limits of integration in (a) we see that the current output depends only on the current and past inputs. Therefore, the system is causal.

2. Consider the system described by the following differential equation:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} - 2y(t) = x(\beta t), \quad \beta > 0, \quad t > 0, \quad y(0) = 0$$

(a) (5 points) Find $h_\tau(t)$, the impulse response of the system.

(b) (5 points) Let $x(t) = \delta(t)$, an impulse function sitting at zero. Find $y(t)$ by applying the Laplace transform on the differential equation above and taking the inverse Laplace transform of the resulting $Y(s)$. Is the answer you got same as (a) with $\tau = 0$?

(c) (5 points) Let $x(t) = \delta(t-\tau)$, an impulse function sitting at τ. Find $y(t)$ by applying the Laplace transform on the differential equation above and taking the inverse Laplace transform of the resulting $Y(s)$. Is the answer you got same as (a)? (Assume $\tau > 0$)

(d) (5 points) Let $x(t) = \mathrm{u}(t-\tau)$. Can we find $y(t)$ by multiplying $X(s) = \mathcal{L}\{x(t)\}$ and $H(s) = \mathcal{L}\{h_\tau(t)\}$ and taking the inverse Laplace transform of $X(s)H(s)$? Why or why not?

Solution:

(a)

$$\mathrm{e}^{-2t}y(t) = \int_0^t \mathrm{e}^{-2\tau}x(\beta\tau)\ \mathrm{d}\tau$$

$$y(t) = \int_{-\infty}^{\infty} \mathrm{e}^{2(t-\tau)}\mathrm{u}(\tau)\mathrm{u}(t-\tau)x(\beta\tau)\ \mathrm{d}\tau$$

Let $\sigma = \beta\tau$. Then $\mathrm{d}\tau = \frac{1}{\beta}\,\mathrm{d}\sigma$.

$$y(t) = \int_{-\infty}^{\infty} \frac{1}{\beta}\mathrm{e}^{2(t-\frac{\sigma}{\beta})}\mathrm{u}\left(\frac{\sigma}{\beta}\right)\mathrm{u}\left(t-\frac{\sigma}{\beta}\right)x(\sigma)\ \mathrm{d}\sigma$$

$$\therefore h_\tau(t) = \frac{1}{\beta}\mathrm{e}^{2(t-\frac{\tau}{\beta})}\mathrm{u}\left(\frac{\tau}{\beta}\right)\mathrm{u}\left(t-\frac{\tau}{\beta}\right)$$

Note: the system is not LTI.

(b) Using the scaling property of Laplace transform on the right side of the equation, we get

$$sY(s) - 2Y(s) = \frac{1}{\beta}$$

$$Y(s) = \frac{1}{\beta}\frac{1}{s-2}, \quad \mathrm{Re}(s) > 2.$$

$$\therefore y(t) = \frac{1}{\beta}\,\mathrm{e}^{2t}\mathrm{u}(t)$$

The answer is same as (a) with $\tau = 0+$.

(c) Because $x(t) = \delta(t-\tau)$, $x(\beta t) = \delta(\beta t - \tau) = \delta\left(\beta\left(t - \frac{\tau}{\beta}\right)\right)$. By applying the scaling and delay properties of the Laplace transform, we have that $X(s) = \frac{1}{\beta}\mathrm{e}^{-s\frac{\tau}{\beta}}$. This yields:

$$sY(s) - 2Y(s) = \frac{\mathrm{e}^{-\frac{\tau}{\beta}s}}{\beta}$$

$$Y(s) = \frac{1}{\beta}\frac{\mathrm{e}^{-\frac{\tau}{\beta}s}}{s-2}, \quad \mathrm{Re}(s) > 2.$$

$$\therefore y(t) = \mathrm{e}^{2(t-\frac{\tau}{\beta})}\mathrm{u}\left(t - \frac{\tau}{\beta}\right),$$

which is the same as (a) when $\tau > 0$.

(d)

$$X(s) = \frac{\mathrm{e}^{-s\tau}}{s}.$$

The Laplace Transform technique cannot be used when the system is not LTI, since we can not express the output of the system as the convolution of the input and the impulse response of the system.

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3. Compute the inverse Laplace transform of $F(s)$, where

$$F(s) = \frac{s^2}{s^2 + 6s + 10}$$

Solution:

$$\begin{aligned} F(s) &= \frac{s^2}{s^2 + 6s + 10} \\ &= 1 + \frac{-6s - 10}{s^2 + 6s + 10} \\ &= 1 - \frac{6s + 18 - 8}{(s+3)^2 + 1} \\ &= 1 - 6\frac{s+3}{(s+3)^2 + 1} + 8\frac{1}{(s+3)^2 + 1} \end{aligned}$$

Therefore,

$$f(t) = \delta(t) - 6\mathrm{e}^{-3t}\cos(t)\mathrm{u}(t) + 8\mathrm{e}^{-3t}\sin(t)\mathrm{u}(t)$$