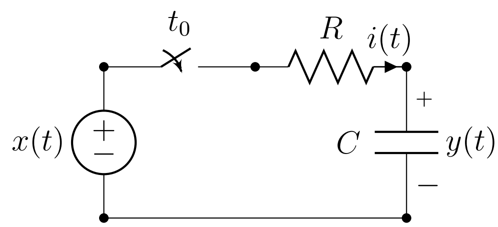
Solutions

1. Consider the system constructed by the following R-C circuit;



where $R = 1\,\Omega$ and $C = 0.2\,\mathrm{F}$. Assume $t_0 = 0$ and $y(0) = 0$.

(a) (5 points) A capacitor has a voltage that is proportional to the charge, which is equal to the integral of the current. And the capacitance C, is defined as the ratio of charge $q(t)$ on each conductor to the voltage $v(t)$ between them: $C = q(t)/v(t)$. Considering these relationships, write a differential equation describing the given system, where $y(t) = v(t)$.

(b) (10 points) What is the impulse response, $h(t;\tau)$, of this system.

(c) (10 points) Is this system linear, time-invariant, causal? Explain your answers.

(d) (5 points) What is the output of the system when the input is $x(t) = 5t\mathrm{u}(t)$? Compute the output in the **time domain**.

Solution:

(a) The voltage-current relationship in a capacitor is as following:

$$v(t) = \frac{1}{C}q(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,\mathrm{d}\tau,$$
$$\Rightarrow i(t) = C\frac{\mathrm{d}}{\mathrm{d}t}v(t).$$

Applying Kirchhoff's voltage law,

$$Ri(t) + y(t) = x(t), \quad i(t) = C\frac{\mathrm{d}}{\mathrm{d}t}y(t).$$

Therefore, the resulting differential equation is,

$$RC\frac{\mathrm{d}}{\mathrm{d}t}y(t) + y(t) = x(t), \quad t > t_0$$

Using the given values,

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 5y(t) = 5x(t), \quad t > 0$$

(b) By multiplying both sides by e^{5t}, the differential equation can be rewritten as

$$\frac{d}{dt}\{e^{5t}y(t)\} = 5e^{5t}x(t)$$

Integrating both sides from 0 to t results in

$$e^{5t}y(t) - e^{5\cdot 0}y(0) = \int_0^t 5e^{5\sigma}x(\sigma)\,d\sigma$$

or

$$y(t) = 5e^{-5t}\int_0^t e^{5\sigma}x(\sigma)\,d\sigma$$

The impulse response $h(t;\tau)$ is the output of the system to the input $\delta(t-\tau)$. Therefore,

$$\begin{aligned}
h(t;\tau) &= 5e^{-5t}\int_0^t e^{5\sigma}\delta(\sigma-\tau)\,d\sigma \\
&= 5e^{-5t}\int_{-\infty}^{\infty} e^{5\sigma}u(\sigma)u(t-\sigma)\delta(\sigma-\tau)\,d\sigma \\
&= 5e^{-5(t-\tau)}u(t-\tau)
\end{aligned}$$

(c) The system is linear since the output is the integration of a linear function of the input. Or, for any k_1, k_2 and x_1, x_2,

$$\begin{aligned}
T[k_1x_1(t) + k_2x_2(t)] &= 5e^{-5t}\int_0^t e^{5\sigma}(k_1x_1(\sigma) + k_2x_2(\sigma))\,d\sigma \\
&= k_1\cdot 5e^{-5t}\int_0^t e^{5\sigma}x_1(\sigma)\,d\sigma + k_2\cdot 5e^{-5t}\int_0^t e^{5\sigma}x_2(\sigma)\,d\sigma \\
&= k_1T[x_1(t)] + k_2[x_2(t)].
\end{aligned}$$

It's time-invariant since $h(t;\tau)$ is only a function of $t-\tau$. It's also verified that $T[x(t-\tau)] = y(t-\tau)$ for any τ.

Moreover, the system is causal as $y(t)$ depends on $x(\sigma)$ for $\sigma \le t$.

(d)

$$\begin{aligned}
y(t) &= 5e^{-5t}\int_0^t e^{5\sigma}[5\sigma u(\sigma)]\,d\sigma \\
&= 5^2e^{-5t}\left(\left[\frac{1}{5}\sigma e^{5\sigma}\right]_{\sigma=0}^{t} - \int_0^t \frac{1}{5}e^{5\sigma}\,d\sigma\right) \\
&= [5e^{-5t}te^{5t} - (e^{5t}-1)]u(t) \\
&= [-e^{5t} + 5t + 1]u(t)
\end{aligned}$$

2. A linear, time-invariant, causal system is described by the following input-output relation

$$y(t) = \int_{t-1}^{t} \mathrm{e}^{-a(t-\sigma)} x(\sigma - 3)\, \mathrm{d}\sigma, \quad -\infty < t < \infty, a \in \mathbb{R}.$$

(a) (5 points) What is the impulse response, $h(t)$, of this system?

(b) (10 points) When the input is $x_1(t) = \mathrm{e}^{-at}\mathrm{u}(t)$, what is the output, $y_1(t)$, of this system ?

(c) (10 points) When the input is $x_2(t) = \cos(2(t-1))\mathrm{u}(t-1)$, what is the output, $y_2(t)$, of this system ?

Solution:

(a) Note that the system is assumed linear, time-invariant and causal, then

$$\begin{aligned} h(t) = T[\delta(t)] &= \int_{t-1}^{t} \mathrm{e}^{-a(t-\sigma)} \delta(\sigma - 3)\, \mathrm{d}\sigma \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{-a(t-\sigma)} \mathrm{u}(\sigma - (t-1))\mathrm{u}(t-\sigma)\delta(\sigma-3)\, \mathrm{d}\sigma \\ &= \mathrm{e}^{-a(t-3)}\mathrm{u}(t-3)\mathrm{u}(4-t) = \mathrm{e}^{-a(t-3)}\left[\mathrm{u}(t-3) - \mathrm{u}(t-4)\right] = \begin{cases} \mathrm{e}^{-a(t-3)} & \text{if } 3 < t < 4, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

.

Or we can extract it from the following expression:

$$y(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-a(t-\sigma)} \mathrm{u}(t-\sigma)\mathrm{u}(\sigma - (t-1)) x(\sigma - 3)\, \mathrm{d}\sigma$$

Setting $\sigma' = \sigma - 3$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} \mathrm{e}^{-a(t-\sigma'-3)} \mathrm{u}(t-\sigma'-3)\mathrm{u}(\sigma'+3-(t-1)) x(\sigma')\, \mathrm{d}\sigma' \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{-a(t-\sigma'-3)} \mathrm{u}(t-\sigma'-3)\mathrm{u}(4-(t-\sigma')) x(\sigma')\, \mathrm{d}\sigma' \\ &= \int_{-\infty}^{\infty} h(t-\sigma') x(\sigma')\, \mathrm{d}\sigma', \end{aligned}$$

hence $h(t) = \mathrm{e}^{-a(t-3)}\mathrm{u}(t-3)\mathrm{u}(4-t)$.

(b)

$$
\begin{aligned}
y_1(t) &= T[\mathrm{e}^{-at}\mathrm{u}(t)] \\
&= \int_{t-1}^{t} \mathrm{e}^{-a(t-\sigma)}\mathrm{e}^{-a(\sigma-3)}\mathrm{u}(\sigma-3)\,\mathrm{d}\sigma \\
&= \mathrm{e}^{-a(t-3)} \int_{t-1}^{t} \mathrm{u}(\sigma-3)\,\mathrm{d}\sigma. \\
&= \begin{cases} 0 & \text{if } t<3, \\ \mathrm{e}^{-a(t-3)} \int_3^t \mathrm{d}\sigma & \text{if } 3<t<4, \\ \mathrm{e}^{-a(t-3)} \int_{t-1}^t \mathrm{d}\sigma & \text{if } t>4, \end{cases} \\
&= \begin{cases} 0 & \text{if } t<3, \\ \mathrm{e}^{-a(t-3)}(t-3) & \text{if } 3<t<4, \\ \mathrm{e}^{-a(t-3)} & \text{if } t>4, \end{cases}
\end{aligned}
$$

(c) We have $x_2(t) = \cos(2(t-1))\mathrm{u}(t-1)$.

Let $x_2(t) = x_3(t-1)$, where $x_3(t) = \cos(2t)\mathrm{u}(t) = \frac{1}{2}(\mathrm{e}^{\mathrm{j}2t} + \mathrm{e}^{-\mathrm{j}2t})\mathrm{u}(t)$.

Also, setting $x_4(t) = \mathrm{e}^{bt}\mathrm{u}(t)$, $x_3(t) = \frac{1}{2}x_4(t)\Big|_{b=2\mathrm{j}} + \frac{1}{2}x_4(t)\Big|_{b=-2\mathrm{j}}$

Then,

$$
\begin{aligned}
y_4(t) = T[x_4(t)] &= \int_{t-1}^{t} \mathrm{e}^{-a(t-\sigma)} x_4(\sigma-3)\,\mathrm{d}\sigma \\
&= \int_{t-1}^{t} \mathrm{e}^{-a(t-\sigma)}\mathrm{e}^{b(\sigma-3)}\mathrm{u}(\sigma-3)\,\mathrm{d}\sigma \\
&= \begin{cases} 0 & \text{if } t<3, \\ \frac{1}{a+b}\left(\mathrm{e}^{b(t-3)} - \mathrm{e}^{-a(t-3)}\right) & \text{if } 3<t<4, \\ \frac{1-\mathrm{e}^{-(a+b)}}{a+b}\left(\mathrm{e}^{b(t-3)}\right) & \text{if } t>4. \end{cases}
\end{aligned}
$$

Since the system is linear,

$$
\begin{aligned}
y_3(t) = T[x_3(t)] &= \frac{1}{2}\left(T[\mathrm{e}^{\mathrm{j}2t}\mathrm{u}(t)] + T[\mathrm{e}^{-\mathrm{j}2t}\mathrm{u}(t)]\right) \\
&= \frac{1}{2}\left(y_4(t)\Big|_{b=-2\mathrm{j}} + y_4(t)\Big|_{b=2\mathrm{j}}\right)
\end{aligned}
$$

(i) if $t<3$, $y_3(t) = 0$,

(ii) if $3 < t < 4$,

$$
\begin{aligned}
y_3(t) &= \frac{1}{2}\left(\frac{1}{a+2\mathrm{j}}(\mathrm{e}^{2\mathrm{j}(t-3)} - \mathrm{e}^{-a(t-3)}) + \frac{1}{a-2\mathrm{j}}(\mathrm{e}^{-2\mathrm{j}(t-3)} - \mathrm{e}^{-a(t-3)})\right) \\
&= \frac{1}{a^2+4}\left(a \cdot \frac{\mathrm{e}^{\mathrm{j}\cdot 2(t-3)} + \mathrm{e}^{-\mathrm{j}\cdot 2(t-3)}}{2} + 2 \cdot \frac{\mathrm{e}^{\mathrm{j}\cdot 2(t-3)} - \mathrm{e}^{-\mathrm{j}\cdot 2(t-3)}}{2\mathrm{j}} - a\mathrm{e}^{-a(t-3)}\right) \\
&= \frac{1}{a^2+4}\left(a\cos(2(t-3)) + 2\sin(2(t-3)) - a\mathrm{e}^{-a(t-3)}\right)
\end{aligned}
$$

(iii) if $t > 4$,

$$
\begin{aligned}
y_3(t) &= \frac{1}{2}\left(\frac{1-\mathrm{e}^{-(a+2\mathrm{j})}}{a+2\mathrm{j}}\mathrm{e}^{2\mathrm{j}(t-3)} + \frac{1-\mathrm{e}^{-(a-2\mathrm{j})}}{a-2\mathrm{j}}\mathrm{e}^{-2\mathrm{j}(t-3)}\right) \\
&= \frac{1}{a^2+4}\left(a \cdot \frac{\mathrm{e}^{\mathrm{j}\cdot 2(t-3)} + \mathrm{e}^{-\mathrm{j}\cdot 2(t-3)}}{2} - a\mathrm{e}^{-a} \cdot \frac{\mathrm{e}^{\mathrm{j}\cdot 2(t-4)} + \mathrm{e}^{-\mathrm{j}\cdot 2(t-4)}}{2}\right. \\
&\quad \left. + 2 \cdot \frac{\mathrm{e}^{\mathrm{j}\cdot 2(t-3)} - \mathrm{e}^{-\mathrm{j}\cdot 2(t-3)}}{2\mathrm{j}} - 2\mathrm{e}^{-a}\frac{\mathrm{e}^{\mathrm{j}\cdot 2(t-4)} - \mathrm{e}^{-\mathrm{j}\cdot 2(t-4)}}{2\mathrm{j}}\right) \\
&= \frac{1}{a^2+4}\left(a\cos(2(t-3)) - a\mathrm{e}^{-a}\cos(2(t-4)) + 2\sin(2(t-3)) - 2\mathrm{e}^{-a}\sin(2(t-4))\right)
\end{aligned}
$$

Also, from the time-invariance of the system,

$$
y_2(t) = T[x_2(t)] = T[x_3(t-1)] = y_3(t-1)
$$

(i) if $t - 1 < 3$, $y_2(t) = 0$,
(ii) if $3 < t - 1 < 4$,

$$
y_2(t) = \frac{1}{a^2+4}\left(a\cos(2(t-4)) - 2\sin(2(t-4)) - a\mathrm{e}^{-a(t-4)}\right)
$$

(iii) if $t - 1 > 4$,

$$
y_2(t) = \frac{1}{a^2+4}\left(a\cos(2(t-4)) - a\mathrm{e}^{-a}\cos(2(t-5)) + 2\sin(2(t-4)) - 2\mathrm{e}^{-a}\sin(2(t-5))\right)
$$

3. Compute the Laplace transform and determine the corresponding region of convergence of the following signals:

(a) (10 points)

$$y_1(t) = \int_0^t \mathrm{e}^{-(21t-\sigma)} \cos(t-\sigma)\mathrm{e}^{-\sigma} \sin(\sigma)\ \mathrm{d}\sigma, \quad t > 0.$$

(b) (10 points)

$$y_2(t) = \int_0^t \sigma \sin(\sigma)\ \mathrm{d}\sigma, \quad t > 0.$$

(c) (10 points)

$$y_3(t) = \mathrm{e}^t (t-5)^2 \mathrm{u}(t-5), \quad t > 0.$$

Solution:

(a)

$$\begin{aligned} y_1(t) &= \int_0^t \mathrm{e}^{-(21t-\sigma)} \cos(t-\sigma)\mathrm{e}^{-\sigma} \sin(\sigma)\, \mathrm{d}\sigma \\ &= \mathrm{e}^{-20t} \int_0^t \mathrm{e}^{-(t-\sigma)} \cos(t-\sigma)\mathrm{e}^{-\sigma} \sin(\sigma)\, \mathrm{d}\sigma \\ &= \mathrm{e}^{-20t} (\mathrm{e}^{-t}\cos(t)) * (\mathrm{e}^{-t}\sin(t)) \end{aligned}$$

Since

$$\mathcal{L}_s\{(\mathrm{e}^{-t}\cos(t)) * (\mathrm{e}^{-t}\sin(t))\} = \frac{s+1}{(s+1)^2+1} \frac{1}{(s+1)^2+1}$$

then

$$Y_1(s) = \frac{s+21}{(s+21)^2+1} \frac{1}{(s+21)^2+1}, \quad \mathrm{Re}(s) > -21$$

(b)

$$\begin{aligned} Y_2(s) &= \frac{1}{s}\mathcal{L}_s\{t\sin(t)\} \\ &= \frac{1}{s}(-1)^1 \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1}\right) \\ &= \frac{1}{s}\frac{2s}{(s^2+1)^2} = \frac{2}{(s^2+1)^2}, \quad \mathrm{Re}(s) > 0 \end{aligned}$$

(c)

$$y_3(t) = \mathrm{e}^5 \mathrm{e}^{(t-5)} (t-5)^2 \mathrm{u}(t-5)$$

We know

$$\mathcal{L}_s\{\mathrm{e}^t t^2 \mathrm{u}(t)\} = \frac{2}{(s-1)^3}$$

Therefore,

$$Y_3(s) = \mathrm{e}^5 \mathrm{e}^{-5s} \frac{2}{(s-1)^3} = \frac{2\mathrm{e}^{-5(s-1)}}{(s-1)^3}, \quad \mathrm{Re}(s) > 1$$

4. (15 points) The following are inputs to a time-invariant system $\mathcal{S}_2$ and the corresponding outputs:

$$x_1(t) = \mathrm{u}(t) - \mathrm{u}(t-2) \;\;\rightarrow\;\; y_1(t) = \mathrm{e}^{-t}\mathrm{u}(t) - \mathrm{e}^{-2t}\mathrm{u}(t-1),$$
$$x_2(t) = \mathrm{u}(t) - \mathrm{u}(t-1) \;\;\rightarrow\;\; y_2(t) = -\mathrm{e}^{-t}\mathrm{u}(t).$$

Can the system said to be linear? Why?

Solution:

$$x_1(t) - x_2(t-1) = \mathrm{u}(t) - \mathrm{u}(t-1) = x_1(t).$$

Assume that the system is linear. As it's also time-invariant, from the prior formula, we should have

$$T[x_1(t) - x_2(t-1)] = T[x_2(t)],$$

i.e.,

$$y_1(t) - y_2(t-1) = y_2(t).$$

However, the left-hand side is

$$\text{LHS} = \mathrm{e}^{-t}\mathrm{u}(t) + (\mathrm{e}^{-(t-1)} - \mathrm{e}^{-2t})\mathrm{u}(t-1),$$

and the right-hand side is

$$\text{RHS} = -\mathrm{e}^{-t}\mathrm{u}(t),$$

which are not equal to each other, a contradiction. So the system is not linear.