Solutions

1. Consider the system, $\mathcal{S}$, described by the differential equation

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 2y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t},$$

with all-zero initial conditions.

(a) (5 points) Write down the system function, $H(s)$.

(b) (5 points) Write down the system's impulse response, $h(t)$.

(c) (5 points) Write down the the output, $y(t)$ when the input is

$$x(t) = \mathrm{e}^{-2t}u(t) + \cos(2t).$$

Solution:

(a) $Y(s)(s+2) = sX(s)$ so $H(s) = \frac{s}{s+2} = 1 - \frac{2}{s+2}$, ROC $= \{\mathrm{Re}(s) > -2\}$.

(b) $h(t) = \delta(t) - 2\mathrm{e}^{-2t}u(t)$.

(c) $x(t) = x_1(t) + x_2(t)$, where $x_1(t) = \mathrm{e}^{-2t}u(t)$, $x_2(t) = \cos(2t)$. $X_1(s) = \frac{1}{s+2}$, $Y_1(s) = \frac{s}{(s+2)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$ with $A = 1$, $B = -2$, therefore $y(t) = \mathrm{e}^{-2t}u(t) + 2t\mathrm{e}^{-2t}u(t)$. $y_2(t) = |H(\mathrm{j}2)|\cos(2t + \angle H(\mathrm{j}2))$. $H(\mathrm{j}2) = \frac{\mathrm{j}2}{\mathrm{j}2+2} = \frac{\mathrm{j}}{\mathrm{j}+1} = \frac{\mathrm{j}(1-\mathrm{j})}{2} = \frac{1+\mathrm{j}}{2}$, $|H(\mathrm{j}2)| = 1/\sqrt{2}$, $\angle H(\mathrm{j}2) = \pi/4$. $y_2(t) = (1/\sqrt{2})\cos(2t + \pi/4)$. $y(t) = y_1(t) + y_2(t)$.

2. Consider a linear, time-invariant system $\mathcal{S}$ with impulse response

$$h(t) = u(t) - u(t-2).$$

(a) (5 points) Write down the system function, $H(s)$, of system $\mathcal{S}$. What is its region of convergence? (Hint: Compute $H(s)|_{s=0}$ separately.)

(b) (5 points) The periodic signal,

$$x(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t-n)$$

is used as input to the system $\mathcal{S}$. What is its period?

(c) (5 points) Compute the coefficients of the Fourier series expansion of $x(t)$ given in (b):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \mathrm{e}^{\mathrm{j}k\omega_0 t}.$$

(d) (5 points) Compute the output $y(t)$ corresponding to the input $x(t)$.

Solution:

(a) $H(s) = \int_0^2 \mathrm{e}^{-st}\,\mathrm{d}t = \frac{1-\mathrm{e}^{-2s}}{s}$, $H(0) = \int_0^2 \mathrm{d}t = 2$, ROC $= \mathbb{C}$. (L'Hôpital is also good.)

(b) Period is $T = 2$, $\omega_0 = \pi$.

(c) $X_k = \frac{1}{2}\int_{0-}^{2-}(\delta(t) - \delta(t-1))\mathrm{e}^{-\mathrm{j}\pi k t}\,\mathrm{d}t = \frac{1}{2}(1-\mathrm{e}^{-\mathrm{j}k\pi})$. $X_{2k} = 0$, $X_{2k+1} = 1$.

(d) $Y_k = X_k H(\mathrm{j}k\pi)$.

$$H(\mathrm{j}k\pi) = \frac{1-\mathrm{e}^{-\mathrm{j}2k\pi}}{\mathrm{j}k\pi} = 0,$$

for $k \neq 0$, and $H(\mathrm{j}0) = 2$, therefore, $Y_k = 0$ for all $k \neq 0$, and $Y_0 = 2X_0 = 0$. This implies that $y(t) = 0$.

3. Consider the function

$$f(t) = u(t)te^{-t}.$$

(a) (5 points) Compute the Laplace transform of $f(t)$, $F(s) = \mathcal{L}\{f(t)\}$, and determine its region of convergence.

(b) (5 points) Can one compute the Fourier transform of $f(t)$, $F(j\omega) = \mathcal{F}\{f(t)\}$, as $F(j\omega) = F(s)|_{s=j\omega}$? Why?

(c) (5 points) Compute the Fourier transform of the function $g(t) = |t|e^{-|t|}$, $-\infty < t < \infty$.

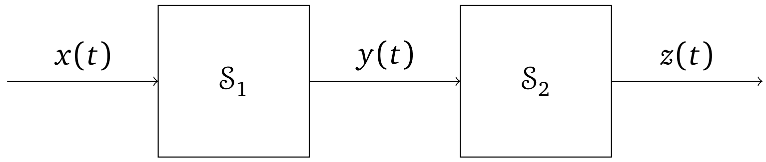
Solution:

(a) $F(s) = -\frac{d}{dt}\frac{1}{s+1} = \frac{1}{(s+1)^2}$, ROC $= \{\mathrm{Re}(s) > -1\}$.

(b) Yes, because the imaginary axis is contained in the ROC.

(c) $g(t) = f(t) + f(-t)$, therefore $G(j\omega) = F(j\omega) + F(-j\omega) = \frac{1}{(j\omega+1)^2} + \frac{1}{(-j\omega+1)^2} = 2\frac{1-\omega^2}{(1+\omega^2)^2}$.

4. The following system



is composed of two stages: The first stage is defined by

$$\mathcal{S}_1: \quad y(t) = [x(t)]^2.$$

The second stage is given by

$$\mathcal{S}_2: \quad z(t) = h_{\text{LP}}(t) \star y(t),$$

where $h_{\text{LP}}(t)$ is an ideal low-pass filter, with frequency function given by

$$H_{\text{LP}}(\text{j}\omega) = u(\omega+1) - u(\omega-1).$$

(a) (5 points) Is $\mathcal{S}_1$ linear? Is it causal? Is $\mathcal{S}_2$ linear? Is it causal? *Briefly* explain your answers.

(b) (5 points) Assume $x(t) = \text{sinc}(t)$. Compute its Fourier transform, $X(\text{j}\omega)$.

(c) (5 points) Compute the Fourier transform of $y(t)$, $Y(\text{j}\omega)$.

(d) (5 points) Compute the Fourier transform of $z(t)$, $Z(\text{j}\omega)$.

(e) (5 points) Write down the expression for $z(t)$.

Solution:

(a) $\mathcal{S}_1$: nonlinear, causal (memory-less). $\mathcal{S}_2$: linear, noncausal ($h_{\text{LP}} \neq 0$ for $t < 0$).

(b) From the table, $X(\text{j}\omega) = \pi \text{rect}\left(\frac{\omega}{2}\right)$.

(c) $Y(\text{j}\omega) = \mathcal{F}\{(\text{sinc}(t))^2\} = \pi \text{tri}\left(\frac{\omega}{2}\right)$.

(d) $Z(\text{j}\omega) = \pi \text{tri}\left(\frac{\omega}{2}\right) H_{\text{LP}}(\text{j}\omega) = \pi \text{tri}\left(\frac{\omega}{2}\right)$ for $-1 < \omega < 1$ and zero elsewhere.

(d) $z(t) = (\text{sinc}(t))^2 \star \text{sinc}(t)$. By inverse-transforming the result in part (d), and using integration by parts, we obtain

$$z(t) = \frac{\sin(t)}{2t} + \frac{1-\cos(t)}{2t^2}.$$

5. Consider the function $f(t)$, $-\infty < t < \infty$, periodic of period $T = 2$, defined by

$$f(t) = \mathrm{e}^{|t|}, \quad -1 < t < 1.$$

(a) (5 points) Do you expect the coefficients, F_n of the Fourier series expansion of $f(t)$,

$$f(t) = \sum_{n=-\infty}^{\infty} F_n \mathrm{e}^{\mathrm{j}n\omega_0 t},$$

to be real or imaginary? Why?

(b) (5 points) Compute the coefficients F_n.

(c) (5 points) Write down the sine-cosine Fourier series expansion of $f(t)$.

(d) (5 points) Compute the power of $f(t)$, $\|f(t)\|^2_{\mathrm{rms}}$.

(e) (5 points) Compute the mean square error, ϵ_1^2, when approximating $f(t)$ up to its first harmonic.

Solution:

(a) Real even, because the function is real and even.

(b) $F_n = \frac{1}{2}\int_{-1}^{1} \mathrm{e}^{|t|}\mathrm{e}^{-\mathrm{j}\pi n t} = \frac{1}{2}\int_{-1}^{0} \mathrm{e}^{(1-\mathrm{j}n\pi)t}\,\mathrm{d}t + \frac{1}{2}\int_{0}^{1} \mathrm{e}^{-(1+\mathrm{j}n\pi)t}\,\mathrm{d}t.$

$$F_n = \frac{1}{2}\left(\frac{(-1)^n\mathrm{e}-1}{1-\mathrm{j}n\pi} + \frac{(-1)^n\mathrm{e}-1}{1-\mathrm{j}n\pi}\right) = \frac{(-1)^n\mathrm{e}-1}{1+n^2\pi^2}.$$

(c) Expansion using $a_n = F_n$, $b_n = 0$.

(d) $\|f(t)\|^2_{\mathrm{rms}} = \sum |F_n|^2 = \frac{1}{2}\int_{-1}^{1} \mathrm{e}^{2|t|}\,\mathrm{d}t = \int_0^1 \mathrm{e}^{2t}\,\mathrm{d}t = \frac{\mathrm{e}^2-1}{2}$

(e) $\epsilon_1^2 = \|f(t)\|^2_{\mathrm{rms}} - (|F_{-1}|^2 + |F_0|^2 + |F_1|^2) = \frac{\mathrm{e}^2-1}{2} - 2\frac{(\mathrm{e}+1)^2}{(1+\pi^2)^2} - (\mathrm{e}-1)^2.$