UCLA
Dept. of Electrical Engineering
EE102: Systems and Signals
Final exam solutions

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1. *The output of a linear, time-invariant, causal system is $y(t) = e^{-2t}\sin(t)U(t)$ when the input is $x(t) = \delta(t) + U(t)$.*

   (a) *Find the transfer function $H(s)$ of the system.*
   $Y(s) = 1/[(s+2)^2+1]$, $X(s) = 1 + 1/s$, $H(s) = Y(s)/X(s) = \frac{s}{[(s+2)^2+1](s+1)}$, with ROC $= \Re\{s\} > -1$.

   (b) *Find the frequency response $H(i\omega)$ of the system. Is $H(i\omega) = H(s)|_{s=i\omega}$? Why?*
   The ROC of $H(s)$ includes the imaginary axis, therefore $H(i\omega) = H(s)|_{s=i\omega} = \frac{i\omega}{[(i\omega+2)^2+1](i\omega+1)}$.

   (c) *The input $x_1(t) = 5 + \cos(t) + e^{-t}U(t)$ is now applied to the system. Compute the corresponding output, $y_1(t)$.*
   We can rewrite $x_1(t) = x_{11}(t) + x_{12}(t) + x_{13}(t)$, where $x_{11}(t) = 5$, $x_{12}(t) = \cos(t)$, $x_{13}(t) = e^{-t}U(t)$. The output is the equal to $y_1(t) = y_{11}(t) + y_{12}(t) + y_{13}(t)$. For $x_{11}(t)$ and $x_{12}(t)$ we can use the fact that the output corresponding to $\cos(\omega_0 t)$ is equal to $|H(i\omega_0)|\cos(\omega_0 t + \theta(\omega_0))$, where $H(i\omega_0) = |H(i\omega_0)|e^{i\theta(\omega_0)}$. We use this expression with $\omega_0 = 0$ for $x_{11}(t)$ and $\omega_0 = 1$ for $x_{12}(t)$, which yield $y_{11}(t) = 0$, $y_{12}(t) = \frac{1}{8}\cos(t)$ ($H(0) = 0$, $H(i) = 1/8$). For $y_{13}(t)$ we can use the Laplace transform: $Y_{13}(s) = H(s)[1/(s+1)] = \frac{s}{[(s+2)^2+1](s+1)^2}$. We have $Y_{13}(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C(s+2)+D}{(s+2)^2+1}$. We obtain $A = 1$, $B = -1/2$, $C = -1$, $D = -1/2$, and $y_{13}(t) = U(t)[e^{-t} - 0.5te^{-t} - e^{-2t}\cos(t) - 0.5e^{-2t}\sin(t)]$.

2. *A linear system $\mathcal{S}_1$ is described by the following input-output relation*

$$y(t) = \int_{-\infty}^{t} 2e^{-(t-1)}\sigma x(\sigma)d\sigma, \quad -\infty < t < \infty,$$

   where $x(t)$ and $y(t)$ are input and output, respectively.

   (a) *Compute the impulse response $h_1(t,\tau)$ of the system $\mathcal{S}_1$. Is $\mathcal{S}_1$ time-invariant? Is it causal? **Briefly** explain your answers.*
   To obtain $h_1(t,\tau)$, we replace $x(t)$ with $\delta(t-\tau)$, which yields

$$h_1(t,\tau) = \int_{-\infty}^{\infty} 2e^{-(t-1)}\sigma\delta(\sigma-\tau)U(t-\sigma)d\sigma = 2e^{-(t-1)}\tau U(t-\tau).$$

   Because $h_1(t,\tau)$ is not solely function of $(t-\tau)$, the system is not TI. Because $h_1(t,\tau) = 0$ for $t < \tau$, the system is causal.

   (b) *Find the output of $\mathcal{S}_1$ when the input is $U(t-1)$.*
   We have

$$y(t) = 2e^{-(t-1)}\int_{-\infty}^{t}\sigma U(\sigma-1)d\sigma = 2e^{-(t-1)}\int_{1}^{t}\sigma d\sigma = e^{-(t-1)}(t^2-1)U(t-1).$$

(c) *The output signal obtained in part (b) is now applied as an input to a linear, time-invariant, causal system $\mathcal{S}_2$. The corresponding output is $z(t) = 2(t-1)^2 e^{-(t-1)}U(t-1)$. Compute the impulse response $h_2(t)$ of system $\mathcal{S}_2$.*
By taking the Laplace transform of $y_1(t) = y(t+1) = e^{-t}(t^2+2t)U(t)$ and $z_1(t) = z(t+1) = 2e^{-t}t^2U(t)$, we have that $H_2(s) = Z_1(s)/Y_1(s) = [4/(s+1)^3]/[2/(s+1)^3 + 2/(s+1)^2] = 2/(s+2)$, which yields $h_2(t) = 2e^{-2t}U(t)$.

3. *The impulse response of a linear, time-invariant system is*

$$h(t) = 2\frac{\sin(2t)}{\pi t}[\cos(6t)]^2.$$

(a) *Find the frequency response of the system, $H(i\omega)$.*
We can rewrite $h(t) = (2/\pi)\,\mathrm{sinc}(2t)(1+\cos(12t))$, which yields $H(i\omega) = \frac{1}{2}\mathrm{rect}((\omega+12)/4) + \mathrm{rect}(\omega/4) + \frac{1}{2}\mathrm{rect}((\omega-12)/4)$.

(b) *Find the output, $y(t)$, when the input is $x(t) = 1 + \sin(2.5t) + \cos(5t)$.*
The signal $x(t)$ has frequency components at frequencies -5,-2.5, 0, 2.5 and 5, but only the component at zero frequency goes through the system, with amplitude given by $H(0) = 1$. The output is $y(t) = 1$.

4. *Consider the signal $x(t)$ periodic of period $T = 2$, defined as $x(t) = t^2$, $-1 < t < 1$. The Fourier sine-cosine series expansion of $x(t)$ is given by*

$$x(t) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi t).$$

(a) *Verify Parseval's theorem for this series, given that $\sum_{n=1}^{\infty} n^{-4} = \pi^4/90$.*
$\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2 dt = \frac{1}{2}\int_{-1}^{1} t^4 dt = 1/5$. This should be equal to $a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty} n^{-4} = \frac{1}{9} + \frac{8}{\pi^4}\frac{\pi^4}{90} = \frac{18}{90} = 1/5$.

(b) *Write the expression for the mean square error ϵ_1^2 when $x(t)$ is approximated by terms up to the first harmonic.*
$\epsilon_1^2 = \|x(t)\|_{\mathrm{RMS}}^2 - a_0^2 - \frac{1}{2}a_1^2 = 1/5 - 1/9 - 8/\pi^4 = 4/45 - 8/\pi^4$.

5. *Consider the following simultaneous differential equations:*

$$\begin{aligned} \frac{dy_1(t)}{dt} + 3y_1(t) + 2.5y_2(t) &= x(t) \\ -2y_1(t) + 2\frac{dy_2(t)}{dt} + 4y_2(t) &= 0. \end{aligned}$$

(a) *Using the Laplace transform, determine the transfer functions relating outputs $y_1(t)$ and $y_2(t)$ to the input $x(t)$ (i.e., $H_1(s) = Y_1(s)/X(s)$ and $H_2(s) =$*

$Y_2(s)/X(s)$).
Taking the Laplace transform of both equation, we obtain:

$$\begin{aligned}(s+3)Y_1(s) + 2.5Y_2(s) &= X(s)\\ -2Y_1(s) + 2(s+2)Y_2(s) &= 0,\end{aligned}$$

from which we have that $Y_1(s) = (s+2)Y_2(s)$ and $Y_2(s)((s+2)(s+3)+2.5) = X(s)$. We then have that $H_2(s) = 1/[(s+2)(s+3)+2.5]$ and $H_1(s) = (s+2)/[(s+2)(s+3)+2.5]$.

(b) *Solve for $y_1(t)$ using the Laplace transform, assuming all initial conditions to be zero and the input $x(t) = U(t)$.*
$Y_1(s) = H_1(s)X(s) = (s+2)/[s(s^2+5s+8.5)] = A/s + (B(s+5/2)+3C/2)/[(s+5/2)^2+9/4]$. We find that $A = 4/17$, $B = -4/17$ and $C = -20/51$, and $y_1(t) = U(t)[A + Be^{-5t/2}\cos(3t/2) + Ce^{-5t/2}\sin(3t/2)]$.