UCLA
Dept. of Electrical Engineering
EE102: Systems and Signals
Final Exam
Friday, June 18, 2004
Please, write your name and ID on the top of each sheet!
Closed book! Calculators are not allowed!

This exam has 4 questions, for a total of 100 points.

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Name and ID: ____________________

Name of person to your left: ____________________

Name of person to your right: ____________________

1. Consider the system $\mathcal{S}$ described by the differential equation

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 4y(t) = \frac{d}{dt}x(t),$$

with all-zero initial conditions.

(a) $\boxed{\text{4 points}}$ Write down the system function, $H(s)$.

Laplace: $Y(s)(s^2+4s+4) = sX(s)$, which means $H(s) = Y(s)/X(s) = \frac{s}{s^2+4s+4} = \frac{s}{(s+2)^2}$.

(b) $\boxed{\text{4 points}}$ Can the frequency response function, $H(i\omega)$, be obtained as $H(i\omega) = H(s)|_{s=i\omega}$? Explain.

Yes, because the poles of $H(s)$, $s = -2$, have negative real part. $H(i\omega) = \frac{i\omega}{(i\omega+2)^2}$.

(c) $\boxed{\text{6 points}}$ Derive the impulse response function, $h(t)$.

$H(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2}$; $B = (s+2)^2 H(s)|_{s=-2} = -2$; $0 = H(0) = \frac{A}{2} - \frac{1}{2}$, hence $A = 1$. Therefore (use the tables!), $h(t) = U(t)\left(e^{-2t} - 2te^{-2t}\right)$.

(d) $\boxed{\text{6 points}}$ Compute the output, $y(t)$, corresponding to the input $x(t) = \cos(t)$.

$y(t) = \frac{1}{2}e^{it}H(i) + \frac{1}{2}e^{-it}H(-i) = \frac{1}{2}e^{it}\frac{i}{(i+2)^2} + \frac{1}{2}e^{-it}\frac{-i}{(-i+2)^2}$, therefore $y_2(t) = \Re\left(e^{it}\frac{i}{(i+2)^2}\right) = \Re\left(e^{it}\frac{i(-i+2)^2}{25}\right) = \Re\left(e^{it}\frac{4+3i}{25}\right)$, hence $y(t) = \frac{4}{25}\cos(t) - \frac{3}{25}\sin(t)$.

2. Consider the signal $x(t)$, periodic of period $T = 2$, defined as $x(t) = \sin\left(\frac{\pi}{2}t\right)$, $-1 < t < 1$.

(a) $\boxed{\text{3 points}}$ Consider its Fourier series coefficients, X_n. Do you expect these coefficients to be purely real or purely imaginary? Why? Answer without explicitly computing the coefficients.

Imaginary, because the function is real odd.

(b) $\boxed{\text{7 points}}$ Compute the coefficients, X_n of the Fourier series expansion of $x(t)$. As a check, verify that the DC value, X_0, has the expected value (what is it?).
$X_n = \frac{1}{2}\int_{-1}^{1} \sin\left(\frac{\pi}{2}t\right) e^{-in\pi t} dt$. Use Euler's formula for the sin(·), and the equalities $e^{\pm i\pi 1/2} = \pm i$, $e^{\pm in\pi} = (-1)^n$. Eventually you'll get $X_n = \frac{i}{\pi}\left[\frac{\sin\left(\frac{\pi}{2}(2n+1)\right)}{2n+1} - \frac{sin\left(\frac{\pi}{2}(2n-1)\right)}{2n-1}\right] = \frac{i4n(-1)^n}{\pi(4n^2-1)}$. $X_0 = 0$, which is the average of the signal.

(c) $\boxed{\text{4 points}}$ What is the mean power of $x(t)$, $\|x(t)\|^2_{\text{RMS}}$?
$\|x(t)\|^2_{\text{RMS}} = \frac{1}{2}\int_{-1}^{1} \sin^2\left(\frac{\pi}{2}t\right) dt = \frac{1}{2}$. You can use the trigonometric equality $\sin^2(x) = \frac{1}{2}[1 - \cos(2x)]$.

(d) $\boxed{\text{4 points}}$ Compute the mean square error, ϵ_1^2, when approximating $x(t)$ up to its first harmonic. You are not required to carry out the calculations, just go as far as you can.
$\epsilon_1^2 = \|x(t)\|^2_{\text{RMS}} - |X_{-1}|^2 - |X_0|^2 - |X_1|^2 = \frac{1}{2} - 2\left(\frac{4}{3\pi}\right)^2 = \frac{1}{2} - \frac{32}{9\pi^2}$.

(e) $\boxed{\text{7 points}}$ The signal $x(t)$ is input to a L, TI, C system with system function $H(s) = \frac{1}{s+1}$. What is the Fourier series expansion of the corresponding output, $y(t)$? No simplifications are required.
$y(t) = \sum_{n=-\infty}^{\infty} Y_n e^{in\pi t}$, with $Y_n = H(in\pi)X_n = \frac{1}{1+in\pi}\frac{i4n(-1)^n}{\pi(4n^2-1)}$.

3. Consider the signal, $x(t)$, whose Fourier transform, $X(i\omega) = |X(i\omega)|e^{i\Theta(\omega)}$, is defined as follows:

$$|X(i\omega)| = \begin{cases} 1, & -3 < \omega < -2 \text{ and } 2 < \omega < 3 \\ 0, & \text{elsewhere} \end{cases} \qquad \Theta(\omega) = \begin{cases} \dfrac{\pi}{2}, & \omega < 0 \\ -\dfrac{\pi}{2}, & \omega > 0 \end{cases}$$

(a) $\boxed{\text{4 points}}$ Write down the mathematical expression for $X(i\omega)$ by using the unit step function, $U(\omega)$.
$X(i\omega) = i\,[U(\omega+3) - U(\omega+2) - U(\omega-2) - U(\omega-3)]$.

(b) $\boxed{\text{3 points}}$ Do you expect $x(t)$ to be a real signal or an imaginary signal? Why? Answer without computing $x(t)$ explicitly.
Real, because the amplitude is even and the phase is odd.

(c) $\boxed{\text{6 points}}$ Derive the expression for $x(t)$. Simplify as much as you can, for example, write all complex exponentials in terms of their real and imaginary parts.
By applying the definition, $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(i\omega)e^{i\omega t}d\omega = \frac{1}{\pi t}[\cos(2t) - \cos(3t)]$.
Alternatively, you can use the fact that $X(i\omega) = i\,\text{rect}(\omega + 2.5) - i\,\text{rect}(\omega - 2.5)$ and use the tables, in which case, you'd get the equivalent expression $x(t) = \frac{1}{2\pi}\,\text{sinc}(t/2)\left(ie^{i2.5t} - ie^{-i2.5t}\right) = -\frac{1}{\pi}\,\text{sinc}(0.5t)\sin(2.5t)$.

(d) $\boxed{\text{6 points}}$ The signal $y(t)$ is obtained by multiplying $x(t)$ with $c(t) = \cos(3t)$. Compute the Fourier transform of $y(t)$, $Y(i\omega)$.
$Y(i\omega) = \frac{1}{2\pi}X(i\omega) * [\pi\delta(\omega+3) + \pi\delta(\omega-3)]$, $Y(i\omega) = \frac{1}{2}X(i(\omega+3)) + \frac{1}{2}X(i(\omega-3))$,
$Y(i\omega) = \frac{i}{2}[U(\omega+6) - U(\omega+5) - U(\omega+1) + 2U(\omega) - U(\omega-1) - U(\omega-5) + U(\omega-6)]$.

(e) $\boxed{\text{6 points}}$ The signal $y(t)$ is now passed through a system with impulse response equal to $h(t) = \frac{3}{\pi}\operatorname{sinc}(3t)$. Compute the output, $z(t)$, of this system.
The system is an ideal low-pass filter with bandwidth $= 3$ rad/s, therefore $Z(i\omega) = \frac{i}{2}[-U(\omega+1) + 2U(\omega) - U(\omega-1)]$. You can either apply the definition of inverse transform, or the fact that $Z(i\omega) = \frac{i}{2}[-\operatorname{rect}(\omega+0.5) + \operatorname{rect}(\omega-0.5)]$. The result is $z(t) = \frac{1}{2\pi t}(\cos(t) - 1) = -\operatorname{sinc}(0.5t)\sin(0.5t)$.

4. The following system

$$x(t) \rightarrow \boxed{\mathcal{S}_1} \rightarrow y(t) \rightarrow \boxed{\mathcal{S}_2} \rightarrow z(t)$$

is composed of two L, TI, C stages.

If the input to the overall system is a unit step function, $x(t) = U(t)$, then the output, $z(t)$, is equal to $z(t) = \left[\cos(t) + 2\sin(t) - e^{-\frac{1}{2}t}\right] U(t)$.

(a) $\boxed{\text{5 points}}$ Compute the overall system function, $H_{12}(s)$.

$H_{12}(s) = \frac{Z(s)}{X(s)} = \frac{\frac{s}{s^2+1} + \frac{2}{s^2+1} - \frac{1}{s+\frac{1}{2}}}{\frac{1}{s}}$, $H_{12}(s) = \frac{5}{2}\frac{s^2}{(s^2+1)(s+\frac{1}{2})}$, with region of convergence $\Re\{s\} > 0$.

(b) $\boxed{\text{5 points}}$ The system $\mathcal{S}_1$ is described by the input output relationship

$$y(t) = x(t) - \frac{1}{2}\int_{-\infty}^{t} e^{-\frac{1}{2}(t-\sigma)}x(\sigma)\,d\sigma.$$

What is the impulse response, $h_1(t)$, of system $\mathcal{S}_1$?
$y(t) = \int_{-\infty}^{t}[\delta(t-\sigma) - \frac{1}{2}e^{-\frac{1}{2}(t-\sigma)}]x(\sigma)\,d\sigma$, therefore $h_1(t) = \delta(t) - \frac{1}{2}e^{-\frac{1}{2}t}U(t)$.

(c) $\boxed{\text{5 points}}$ Given the answers to parts (a) and (b), compute the system function, $H_2(s)$, of system $\mathcal{S}_2$.
$H_{12}(2) = H_1(s)H_2(s)$, therefore $H_2(s) = \frac{H_{12}(s)}{H_1(s)}$ where $H_1(s) = 1 - \frac{1}{2}\frac{1}{s+\frac{1}{2}} = \frac{s}{s+\frac{1}{2}}$. Hence, $H_2(s) = \frac{5}{2}\frac{s}{s^2+1}$.

(d) $\boxed{\text{7 points}}$ Compute the frequency response function, $H_2(i\omega)$, of system $\mathcal{S}_2$.
From the above, $h_2(t) = \frac{5}{2}U(t)\cos(t)$, therefore $H_2(i\omega) = \frac{5}{2}\frac{i\omega}{1-\omega^2} + \frac{5\pi}{4}\delta(\omega+1) + \frac{5\pi}{4}\delta(\omega-1)$.

(e) $\boxed{\text{8 points}}$ Consider now signal $v(t) = e^{-|t|}$, $-\infty < t < \infty$. If $v(t)$ is now the input to system $\mathcal{S}_2$,

$$v(t) \;\rightarrow\; \boxed{\mathcal{S}_2} \;\rightarrow\; w(t),$$

what is the Fourier transform of the corresponding output, $w(t)$?

$V(i\omega) = \frac{2}{1+\omega^2}$, therefore $W(i\omega) = V(i\omega)H_2(i\omega) = \frac{5i\omega}{1-\omega^4} + \frac{5\pi}{4}\delta(\omega+1) + \frac{5\pi}{4}\delta(\omega-1)$.