UCLA Electrical Engineering Dept.
EE102: Systems and Signals
Final Solutions

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1. *Consider a linear system with impulse response $h(t) = e^{-t}(\cos(t) - \sin(t))U(t)$.*

   (a) *Find the transfer function $H(s)$ of the system.*
   From the tables,

$$H(s) = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1} = \frac{s}{(s+1)^2+1}.$$

   (b) *Compute the input $x(t)$ that generates the output $y(t) = e^{-t}\cos(t-1)U(t-1)$.*
   Because $y(t) = e^{-1}e^{(t-1)}\cos(t-1)U(t-1)$, $Y(s) = e^{-1}e^{-st}\frac{s+1}{(s+1)^2+1}$. From the relation $Y(s) = H(s)X(s)$, we have that

$$X(s) = e^{-1}e^{-s}\frac{s+1}{s} = e^{-1}e^{-s}\left(1 + \frac{1}{s}\right).$$

   By taking the inverse Laplace transform, we have that $x(t) = e^{-1}(\delta(t-1) + U(t-1))$.

   (c) *Find the frequency response $H(i\omega)$ of the system. Is $H(i\omega) = H(s)|_{s=i\omega}$?*
   The poles of $H(s)$ are to the left of the imaginary axis, and $h(t) = 0$, $t < 0$, therefore $H(i\omega) = H(s)|_{s=i\omega}$, and

$$H(i\omega) = \frac{i\omega}{(1+i\omega)^2+1} = \frac{i\omega}{(2-\omega^2)+i2\omega}.$$

   (d) *What are the expressions for $|H(i\omega)|$ and $\theta(\omega)$, where $H(i\omega) = |H(i\omega)|e^{i\theta(\omega)}$? Compute $|H(i\sqrt{2})|$ and $\theta(\sqrt{2})$.*

$$|H(i\omega)| = \frac{\omega}{\sqrt{(2-\omega^2)^2+4\omega^2}} = \frac{w}{\sqrt{w^4+4}},$$

$$\theta(\omega) = \begin{cases} \frac{\pi}{2} - \tan^{-1}\left(\frac{2|\omega|}{\omega^2-2}\right), & \omega < -\sqrt{2} \\ 0, & \omega = -\sqrt{2} \\ -\frac{\pi}{2} + \tan^{-1}\left(\frac{2|\omega|}{2-\omega^2}\right), & -\sqrt{2} < \omega < 0 \\ 0, & \omega = 0 \\ \frac{\pi}{2} - \tan^{-1}\left(\frac{2\omega}{2-\omega^2}\right), & 0 < \omega < \sqrt{2} \\ 0, & \omega = \sqrt{2} \\ -\frac{\pi}{2} + \tan^{-1}\left(\frac{2\omega}{\omega^2-2}\right), & \sqrt{2} < \omega \end{cases}$$

   For $\omega = \sqrt{2}$, we have

$$|H(i\sqrt{2})| = \frac{1}{2}, \quad \theta(\sqrt{2}) = \angle\frac{i\sqrt{2}}{i2\sqrt{2}} = 0.$$

(e) *Is it true that for this system the output to the input* $x_1(t) = \cos(\omega_0 t)$ *is given by* $y_1(t) = \Re\{e^{i\omega_0 t} H(i\omega_0)\}$*?*
In general, the output to $x_1(t) = \cos(\omega_0 t) = \frac{1}{2}e^{-i\omega_0 t} + \frac{1}{2}e^{i\omega_0 t}$ is equal to $y_1(t) = \frac{1}{2}H(-i\omega_0)e^{-i\omega_0 t} + \frac{1}{2}H(i\omega_0)e^{i\omega_0 t}$. In our case,

$$H(-i\omega) = \overline{H(i\omega)},$$

so we have that $y_1(t) = 2\Re\{\frac{1}{2}H(i\omega_0)e^{i\omega_0 t}\} = \Re\{H(i\omega_0)e^{i\omega_0 t}\}$.

(f) *The input* $x_2(t) = 100+\cos(2t)$ *is now applied to the system. What is the average (DC) value of the output?* The average DC component of the output, i.e., the 0th Fourier coefficient of the output, is given by $100H(i0) = 0$.

(g) *For the same input* $x_2(t)$ *of the previous question, what is the mean square error of the output when it is approximated up to the third harmonic?*
The signal $x_2(t)$ has a non-zero DC plus one harmonic. Therefore the output will have a DC (actually equal to zero, from the previous question) and one harmonic, at the same frequency, 2 rad/s. If we approximate the output up to the third harmonic, the mean square error will be zero.

(h) *What is the output* $y_3(t)$ *when the input is* $x_3(t) = \cos(\sqrt{2}t) + U(t)$*.*
From a previous question, the output to $x_{31}(t) = \cos(\sqrt{2}t)$ is equal to $y_{31}(t) = \Re\{H(i\sqrt{2})e^{i\sqrt{2}}\} = \frac{1}{2}\cos(\sqrt{2}t)$. The output to $x_{32}(t) = U(t)$ can be obtained by inverse Laplace transforming

$$\frac{1}{s}H(s) = \frac{1}{(s+1)^2+1},$$

which yields $y_{32}(t) = U(t)e^{-t}\sin(t)$. The complete output is $y_3(t) = y_{31}(t) + y_{32}(t) = \frac{1}{2}\cos(\sqrt{2}t) + U(t)e^{-t}\sin(t)$.

2. *Consider the two linear, time-invariant systems whose frequency functions are given by:*

$$H_1(i\omega) = \mathcal{F}\{h_1(t)\} = e^{-i\omega/2}\,\mathrm{rect}\left(\frac{\omega}{2\pi}\right),$$
$$H_2(i\omega) = \mathcal{F}\{h_2(t)\} = -i\,\mathrm{sgn}(\omega)\,\mathrm{rect}\left(\frac{\omega}{2\pi}\right),$$

where $\mathrm{rect}(\omega/\omega_0) = U(\omega+\omega_0/2) - U(\omega-\omega_0/2)$*, and* $\mathrm{sgn}(\omega) = 2U(\omega) - 1$.

(a) *Sketch the amplitude and phase spectra of $H_1(i\omega)$ and $H_2(i\omega)$.*

$$|H_1(i\omega)| = \begin{cases} 1, & -\pi < \omega < \pi \\ 0, & \text{elsewhere} \end{cases}, \quad \theta_1(\omega) = \begin{cases} -\omega/2, & -\pi < \omega < \pi \\ \text{undefined, or } 0, & \text{elsewhere} \end{cases}$$

$$|H_2(i\omega)| = |H_1(i\omega)|, \quad \theta_2(\omega) = \begin{cases} \pi/2, & -\pi < \omega < 0 \\ -\pi/2, & 0 < \omega < \pi \\ \text{undefined or } 0, & \text{elsewhere} \end{cases}$$

(b) *Compute $h_1(t)$ and $h_2(t)$.* From the tables, $h_1(t) = \text{sinc}(\pi(t-1/2))$. As for $h_2(t)$,

$$h_2(t) = \frac{i}{2\pi}\int_{-\pi}^{0} e^{i\omega t} d\omega - \frac{i}{2\pi}\int_{0}^{\pi} e^{i\omega t} d\omega = \frac{1}{2\pi t}(1-e^{-i\pi t}) - \frac{1}{2\pi t}(e^{i\pi t}-1) = \frac{1}{\pi t}(1-\cos(\pi t)).$$

Note that $h_2(0) = 0$.

(c) *Compute the output $y(t)$ of the cascade*

$$x(t) \rightarrow \boxed{H_1(i\omega)} \rightarrow \boxed{H_2(i\omega)} \rightarrow y(t)$$

when the input is $x(t) = \cos(\pi t/2) + 10\sin(2\pi t)$.
Note that $H_{12}(i2\pi) = 0$, so $y(t) = \Re\{H_{12}(i\pi/2)e^{i\pi t/2}\} = \Re\{e^{-i\pi/4}e^{-i\pi/2}e^{i\pi t/2}\} = \cos(\pi t/2 - 3\pi/4)$.

3. *(Poisson summation formula.) Consider the generic function $f(t)$, with Fourier transform $F(i\omega)$. Define the function*

$$g(\theta) = \sum_{n=-\infty}^{\infty} f\left(\frac{\theta}{2\pi} + n\right).$$

(a) *Show that $g(\theta)$ is a periodic function and find its period, T, and $\omega_0 = 2\pi/T$.*
By the definition, $g(\theta + 2\pi) = \sum_{n=-\infty}^{\infty} f(\theta/(2\pi) + 1 + n) = g(\theta)$, by a change of indices $m = n+1$. Therefore $T = 2\pi$ and $\omega_0 = 1$.

(b) *Write $g(\theta)$ as $g(\theta) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega_0\theta}$, and compute the Fourier series coefficients α_n. [Hint: you can exchange integral and summation.]*
By the definition of Fourier series coefficients,

$$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-in\theta}d\theta = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi} f\left(\frac{\theta}{2\pi}+m\right)e^{-in\theta}d\theta$$
$$= \sum_{m=-\infty}^{\infty}\int_{-1/2+m}^{1/2+m} f(\phi)e^{-i2\pi n\phi}d\phi = \int_{-\infty}^{\infty} f(\phi)e^{-i2\pi n\phi}d\phi = F(i2\pi n).$$

(c) *By using the results above, prove Poisson's summation formula,*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} F(i2\pi n).$$

[Hint: consider $g(0)$.]
By definition, $g(0) = \sum_{n=-\infty}^{\infty} f(n)$. By the Fourier series expansion, we have that $g(\theta) = \sum_{n=-\infty}^{\infty} F(i2\pi n)e^{in\theta}$, which yields $g(0) = \sum_{n=-\infty}^{\infty} F(i2\pi n)$.