**F12-EE102-FINAL-PREP**

**A Past Final**

**QUESTION 1** (40%)
(+HINTS)

**(i)** A system $S$ is described by the IPOP relation:

$$\mathbf{y}(\mathbf{t}) = \mathbf{x}(\mathbf{t}) - \int_{-\infty}^{\mathbf{t}} \mathbf{e}^{-(\mathbf{t}-\tau)} \sin(\mathbf{t}-\tau)\, \mathbf{x}(\tau)\, \mathbf{d}\tau, \quad \mathbf{t} > -\infty$$

$$\forall\, \mathbf{t} > -\infty : \qquad \mathbf{x}(\mathbf{t}) \longrightarrow [\, \mathbf{S}\, ] \longrightarrow \mathbf{y}(\mathbf{t})$$

**(a)** Show relationships, if any, between the OP:

$$y_1(t) := \int_{-\infty}^{\infty} \delta(t-\tau) U(\tau) d\tau - \int_0^t e^{-(t-\tau)} \sin(t-\tau)\, d\tau, \quad t \geq 0$$

and the OP:

$$y_2(t) := \int_{-\infty}^{\infty} \delta(t-\tau)\, \delta(\tau) d\tau - \int_{-\infty}^{t} e^{-(t-\tau)} \sin(t-\tau)\, \delta(\tau)\, d\tau, \quad t \geq 0$$

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We have

$$U(t) \longrightarrow [S] \longrightarrow y_1(t) = g(t) \;\leftarrow\; \text{USR}$$

and

$$\delta(t) \longrightarrow [S] \longrightarrow y_2(t) = h(t) \;\leftarrow\; \text{IRF}$$

$$\Rightarrow \qquad h(t) = \frac{dg(t)}{dt}$$

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**(b)** Find the **FRF** (Frequency Response Function) $H(i\omega)$ of $S$.

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We have

$$H(s) = 1 - \frac{1}{(s+1)^2+1} 1 = \frac{(s+1)^2}{(s+1)^2+1}, \quad \mathcal{R}e\, s > -1 \Rightarrow i\omega \in DOC$$

Therefore

$$H(i\omega) := H(s)|_{s=i\omega} = \frac{(1+i\omega)^2}{1+(1+i\omega)^2}$$

**(c)** Let

$$\mathbf{x(t)} = \cos \mathbf{t}, \quad \mathbf{t} \in \mathbf{R}$$

be the IP to $S$ and let $y(t)$ be the corresponding OP:

$$\forall\, \mathbf{t} > -\infty : \qquad \mathbf{x(t)} = \cos \mathbf{t} \;\longrightarrow [\;\mathbf{S}\;] \longrightarrow\; \mathbf{y(t)}$$

Find $Y(i\omega) := \mathcal{F}\{y(t)\}$ then compute $y(t)$ by taking $\mathcal{F}^{-1}\{Y(i\omega)\}$.
Describe another method of finding $y(t)$ — but do not recompute $y(t)$.

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We have

$$\begin{aligned} Y(i\omega) &= H(i\omega)\,\mathcal{F}\{cost\} \\ &= H(i\omega)\,\pi\{\delta(\omega+1)+\delta(\omega-1)\} \\ \Rightarrow \quad y(t) &= \mathcal{F}^{-1}\{Y(i\omega)\} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} Y(i\omega)\,d\omega \\ &= \frac{3}{5}\cos t + \frac{1}{5}\sin t, \quad t \in \mathbf{R} \end{aligned}$$

<u>Second Method</u>

$$\begin{aligned} y(t) &:= y_1(t) + y_2(t) \\ & \frac{1}{2}\,e^{i\omega t} \to [S] \to y_1(t), \quad t \in \mathbf{R} \\ & \frac{1}{2}\,e^{-i\omega t} \to [S] \to y_2(t), \quad t \in \mathbf{R} \end{aligned}$$

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**(ii)** Compute the signal $f(t)$ whose Fourier Transform $F(i\omega)$ is given in terms of its **amplitude**:

$$\begin{aligned} |F(i\omega)| &= -\omega, \quad \text{for} \quad -1 \le \omega < 0, \\ &= 1, \quad \text{for} \quad 0 \le \omega < 1, \\ &= 0, \quad \text{otherwise,} \end{aligned}$$

and its **phase**:

$$
\begin{aligned}
\Theta(\omega) &= \frac{\pi}{2}, \quad \omega < 0, \\
&= -\frac{\pi}{2}, \quad \omega \geq 0.
\end{aligned}
$$

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$$
\begin{aligned}
f(t) &= \frac{-i}{2\pi} \int_{-1}^{0} e^{i\omega t} \omega \, d\omega \\
&= \frac{1}{2\pi} \{ e^{-it} [\frac{1}{t} + i\frac{1}{t^2}] - \frac{i}{t^2} \}, \quad t \in \mathrm{R}
\end{aligned}
$$

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(iii) Let $f(t)$ be a periodic signal with period T, then you know that $f(t)$ admits the Complex Exponentials Fourier Series Expansion:

$$
f(t) = \sum_{n=-\infty}^{\infty} F_n e^{in\omega_0 t}, \quad \omega_0 := \frac{2\pi}{T}
$$

where $F_n, \ n = 0, \pm 1, \pm 2, \ldots$, are the Fourier Coefficients of $f(t)$.

If $F_n$ are real, what can you conclude about $f(t)$?

If $F_n$ and $f(t)$ are both real, what can you conclude about $f(t)$?

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If $F_n$ are real then $f(t)$ admits the FS:

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} F_n \cos n\omega_0 t + i \sum_{n=-\infty}^{\infty} F_n \sin n\omega_0 t \\
&= f_{even}(t) + i f_{odd}(t)
\end{aligned}
$$

If $f(t)$ and $F_n$ are real then $f(t)$ admits the FS:

$$
\begin{aligned}
f(t) &= F_0 + \sum_{n=1}^{\infty} 2F_n \cos n\omega_0 t \qquad (1) \\
&= f(-t)
\end{aligned}
$$

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**(iv)** Find $y(t)$ given that

$$\forall\, \mathbf{t} \in (-\infty, \infty): \quad \cos(\mathbf{5\,t} + \frac{\pi}{\mathbf{2}}) \;\longrightarrow [\;\mathbf{S}:\; \mathbf{H}(\mathbf{i}\omega)\;] \longrightarrow\; \mathbf{y}(\mathbf{t}) = ?$$

where $S$ is a LTI system with FRF $H(i\omega)$.

What would your answer be if, in addition, the system $S$ is also Real[1]?

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$$\begin{aligned} y(t) &:= y_1(t) + y_2(t) \\ y_1(t) &:= \leftarrow [S] \leftarrow \frac{1}{2} e^{(5t+\frac{\pi}{2})} \\ y_2(t) &:= \leftarrow [S] \leftarrow \frac{1}{2} e^{-(5t+\frac{\pi}{2})} \end{aligned}$$

Therefore

$$y(t) = \frac{1}{2}\{H(i5)e^{(5t+\frac{\pi}{2})} + H(-i5)e^{-(5t+\frac{\pi}{2})}\}$$

If $S$ is a Real System then

$$\overline{H(i\omega)} = H(-i\omega) \quad \Rightarrow \quad y(t) = \mathcal{R}e\{H(i\omega)\, e^{(5t+\frac{\pi}{2})}\}$$

[1] that is its IRF $h(t)$ is real

**QUESTION 2 (30%)**

Consider the cascaded combination $S_{12}$ of LTIC systems $S_1$ and $S_2$:

$$\mathbf{x(t)} \rightarrow [\mathbf{S_1} : \mathbf{LTIC}] \rightarrow [\mathbf{S_2} : \mathbf{LTIC}] \rightarrow \mathbf{z(t)}$$

where

$$\mathbf{x(t)} = \mathbf{e^{-t}\,U(t)}$$

and

$$\mathbf{z(t)} = \left[\cos\mathbf{t} + \sin\mathbf{t} - \mathbf{e^{-t}}\right]\mathbf{U(t)}$$

**(i)** Compute the IRF $h_{12}(t)$ of the cascaded system $S_{12}$.

Now suppose that $S_1$ is described by the IPOP relation:

$$\mathbf{x(t)} \rightarrow [\mathbf{S_1} : \mathbf{LTIC}] \rightarrow \mathbf{y(t)} = \int_{-\infty}^{\mathbf{t}} \mathbf{x}(\sigma)\mathbf{d}\sigma, \quad \mathbf{t} > -\infty$$

Your problem is to write down the System Function $H_2(s)$ and the IRF $h_2(t)$ of $S_2$.

**(ii)**) Derive the FRF $H_2(i\omega)$ of system $S_2$ — from its System Function $H_2(s)$ — if possible. If it is NOT possible find $H_2(i\omega)$ by "your" method.

**(iii)** Consider again:

$$\mathbf{x(t)} \rightarrow [\mathbf{S_1} : \mathbf{LTIC}] \rightarrow \mathbf{y(t)} = \int_{-\infty}^{\mathbf{t}} \mathbf{x}(\sigma)\mathbf{d}\sigma, \quad \mathbf{t} > -\infty$$

where

$$\begin{aligned} x(t) &= e^{t}, \quad t < 0 \\ &= e^{-t}, \quad t \geq 0 \end{aligned}$$

Your problem is to find the output $y(t)$ — by any method which you are most comfortable with.

**QUESTION 3 (30%)**

**(i)** Let $x(t)$ be the periodic signal defined over one period by

$$x(t) = \begin{cases} \sin(\pi t), & -\frac{1}{2} \le t \le \frac{1}{2}, \\ 0, & -1 \le t < -\frac{1}{2} \text{ and } \frac{1}{2} < t \le 1. \end{cases}$$

Plot $x(t)$, then write down the Fourier Sine-Cosine series representation of $x(t)$. Then compute the MSE (Mean Square Error) when $x(t)$ is approximated by the finite series of the form

$$\widehat{x}(t) = \sum_{n=-1}^{1} X_n \, e^{in\omega_0 t}. \qquad (\star)$$

**(ii)** The periodic signal $x(t)$ of Part (i) is now applied to a LTIC system $S$ whose IRF is $h(t)U(t)$

$$\mathbf{x(t)} \longrightarrow [\mathbf{S} : \mathbf{h(t)U(t)}] \longrightarrow \mathbf{y(t)}$$

a) Write down the Fourier Series representation of $y(t)$ — assuming that $\mathbf{h(t)}\,\mathbf{U(t)}$ is known.

b) Now let $x(t)$ be approximated by $\widehat{x}(t)$ as in $(\star)$ and consider

$$\widehat{\mathbf{x}}\mathbf{(t)} \longrightarrow [\mathbf{S} : \mathbf{h(t)U(t)}] \longrightarrow \widehat{\mathbf{y}}\mathbf{(t)}$$

Can you conclude that $\widehat{y}(t)$ is an approximation of $y(t)$ — in the same way as $\widehat{x}(t)$ is that of $x(t)$? If so write down an expression for the MSE

$$\overline{\mathcal{E}_1^2}(y)$$

c) Under what conditions would you have

$$\overline{\mathcal{E}_1^2}(x) = \overline{\mathcal{E}_1^2}(y)?$$

The End

Happy Holidays To All