Part I: Time-Domain Analysis

(Do NOT use Laplace Transforms here)

1. The IPOP relation of a SISO system S is:

$$x(t) \longrightarrow [S] \longrightarrow y(t)$$

u(t-τ)

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}\, x(\tau)\, d\tau, \quad t \in (-\infty, \infty)$$

Write down the IRF $h(t,\tau)$ of S. Then compute its output $y(t)$ given that its input $x(t)$ is

$$x(t) = e^{-(t-1)}\, U(t-1) + U(2-t), \quad t \in (-\infty, \infty)$$

2. Let system S_1 be described by the IPOP relation:

$$x(t) \longrightarrow [S_1] \longrightarrow y(t), \quad y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau, \quad t \in (-\infty, \infty)$$

and let system S_2 be described by the IPOP relation:

$$v(t) \longrightarrow [S_2] \longrightarrow w(t)$$

$$w(t) = v(t) - \int_{-\infty}^{\infty} \sin(t-\tau)\, U(t-\tau)\, v(\tau)\, d\tau, \quad t \in (-\infty, \infty)$$

Compute the IRF $h_{21}(t)$ of the cascaded system $S_{21} := S_2\, S_1$.

3. Let S be a L, TI, C system with IRF $h(t)U(t)$. Let $y(t)$ be the output given by:

$$y(t) = \int_{4}^{\infty} h(t-\tau)\, U(t-\tau)\, d\tau, \quad t \geq 4.$$

Find the input $x(t)$ which results in the output $y(t)$. Finally, would you say that the IRF of S is

$$h(t) = \frac{dy(t+4)}{dt}, \quad t \geq 0,$$

where $y(t)$ is as given? Please explain your answer.

Part II: s-Domain Analysis

(This is where Laplace Transforms shine)

4. Given the following information regarding a system S:

$$e^{-t}\,U(t) \longrightarrow [\mathbf{S}:\mathbf{L},\mathbf{TI},\mathbf{C}] \longrightarrow e^{-t}(1-t)\,U(t)$$

Your problem is: (i) Find the IRF of S and, (ii) Find an IP $x(t)$ to S so that:

$$x(t)(\text{To Be Found}) \longrightarrow [\mathbf{S}:\mathbf{L},\mathbf{TI},\mathbf{C}] \longrightarrow \cos t\,U(t)$$

5. (i) The Laplace Transform $Y(s)$ of $y(t)$ is

$$Y(s) = \frac{s-4}{s^2+5s+4}.$$

Suppose now that $y(t)$ is the OP of a L, TI, C system S when an appropiate input $x(t)$ is applied to S. Your problem is to characterize the system S as well as the signal $x(t)$.

(ii) Let $F(s)$ be the Laplace Transform of $f(t)$, then

$$\mathcal{L}_s\{\int_0^t e^{-2\tau}\,\tau\, f(\tau)\,d\tau\} = ?$$

6. (i) Express the signal

$$f(t) = (\cos t + \sin t - e^{-t})\,U(t)$$

as a convolution integral.

(ii) Let S be a L,TI,C system which is such that:

$$\sin t\,U(t) \longrightarrow [\mathbf{S}:\mathbf{L},\mathbf{TI},\mathbf{C}] \longrightarrow t\,\cos t\,U(t)$$

Write down the Differential Equation relating an IP $x(t)$ — of S — and its corresponding OP $y(t)$.

For the same system S find $y(t)$ given:

$$\sinh t\,U(t) \longrightarrow [\mathbf{S}:\mathbf{L},\mathbf{TI},\mathbf{C}] \longrightarrow \mathrm{y(t)} = ?$$

Part III: Time-Domain and s-Domain

(Use whatever method(s) which you are most comfortable with)

7. A system S :

$$x(t) \longrightarrow [S] \longrightarrow y(t)$$

is described by the differential equation:

$$\begin{aligned} \frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) &= x(t), \quad t > 0, \\ y(0) &= 0 = \dot{y}(0). \end{aligned}$$

(i) Find the IRF $h(t)$ of the system.

(ii) Can this system be realized by cascading two L, TI, C, systems? If so write down the IRF of the two systems.

8. (i) True or False:

$$\mathcal{L}_s\{ \int_0^t e^{-t} \cos(t-\tau) \cos\tau \, d\tau \} = \frac{(s+1)^2}{[(s+1)^2+1]^2} \; ?$$

Please explain why.

(ii) Let

$$h(t) = e^{-t} \cos t \, U(t)$$

be the IRF of a L, TI, C, system S. Find the Laplace Transform $Y(s)$ of the OP $y(t)$ of S, given that its IP $x(t)$ is

$$x(t) = e^{-t} \sin t \, U(t).$$

9. The IP

$$x(t) = \delta(t) - 2e^{-2t} \, U(t)$$

to a L, TI, C, system S results in the OP $y(t)$:

$$x(t) = \delta(t) - 2e^{-2t} \, U(t) \longrightarrow [\textbf{S: L, TI, C}] \longrightarrow y(t).$$

Moreover, the Laplace Transform $Y(s)$ of $y(t)$ is

$$Y(s) := \mathcal{L}_s\{y(t)\} = \frac{2s}{s^3 + s^2 - 4s - 4}.$$

Your problem is to write down an IPOP relation — in the time-domain — for the system S.

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1. $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau)\,d\tau$

$= \int_{-\infty}^{\infty} e^{-(t-\tau)} U(t-\tau)\, x(\tau)\,d\tau$

Then, $h(t,\tau) = e^{-(t-\tau)} U(t-\tau)$.

$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau)\,d\tau$

$= \int_{-\infty}^{t} e^{-(t-\tau)} [e^{-(\tau-1)} U(\tau-1) + U(2-\tau)]\,d\tau$

$= \int_{-\infty}^{t} e^{-(t-1)} U(\tau-1)\,d\tau$

$+ \int_{-\infty}^{t} e^{-(t-\tau)} U(2-\tau)\,d\tau$

$= (\int_{1}^{t} e^{-(t-1)} d\tau)\, U(t-1)$

$+ \int_{-\infty}^{\min(2,t)} e^{-(t-\tau)}\,d\tau$

$= (\int_{1}^{t} e^{-(t-1)} d\tau)\, U(t-1)$

$+ (\int_{-\infty}^{2} e^{-(t-\tau)} d\tau)\, U(t-2)$

$+ (\int_{-\infty}^{t} e^{-(t-\tau)} d\tau)\, U(2-t)$

$= e^{-(t-1)} [\tau|_1^t]\, U(t-1)$

$+ e^{-t} [e^{\tau}|_{-\infty}^{2}]\, U(t-2)$

$+ e^{-t} [e^{\tau}|_{-\infty}^{t}]\, U(2-t)$

$= (t-1) e^{-(t-1)} U(t-1)$

$+ e^{-(t-2)} U(t-2) + U(2-t)$.

Or

$$= \begin{cases} 1, & t \le 1 \\ (t-1)e^{-(t-1)} + 1, & 1 \le t \le 2, \\ (t-1)e^{-(t-1)} + e^{-(t-2)}, & t \ge 2. \end{cases}$$

2. S_1: $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

$= \int_{-\infty}^{\infty} U(t-\tau)\, x(\tau)\,d\tau$

$h_1(t,\tau) = U(t-\tau)$

Then, S_1 is L, TI and C.

$h_1(t) = U(t)$

S_2: $w(t) = v(t) - \int_{-\infty}^{\infty} \sin(t-\tau) U(t-\tau) v(\tau)\,d\tau$

$= \int_{-\infty}^{\infty} \delta(t-\tau) v(\tau)\,d\tau$

$- \int_{-\infty}^{\infty} \sin(t-\tau) U(t-\tau) v(\tau)\,d\tau$

$= \int_{-\infty}^{\infty} [\delta(t-\tau) - \sin(t-\tau) U(t-\tau)]$

$\cdot v(\tau)\,d\tau$

Then, $h_2(t,\tau) = \delta(t-\tau) - \sin(t-\tau) U(t-\tau)$.

Hence, S_2 is L, TI and C.

$h_2(t) = \delta(t) - \sin(t)\, U(t)$

S_{21}: $h_{21}(t) = \int_{-\infty}^{\infty} h_1(t-\tau) h_2(\tau)\,d\tau$

$= \int_{-\infty}^{\infty} U(t-\tau) [\delta(\tau)$

$- \sin\tau\, U(\tau)]\,d\tau$

$= \int_{-\infty}^{\infty} U(t-\tau) \delta(\tau)\,d\tau$

$- \int_{-\infty}^{\infty} \sin\tau\, U(t-\tau) U(\tau)\,d\tau$

$= U(t) - [\int_0^t \sin\tau\,d\tau]\, U(t)$

$= U(t) - [\cos t - 1]\, U(t)$

$= \cos t\, U(t)$

3. $y(t) = \int_{4}^{\infty} h(t-\tau) U(t-\tau)\,d\tau$

$= \int_{-\infty}^{\infty} h(t-\tau) U(t-\tau) U(\tau-4)\,d\tau$

Thus, $x(\tau) = U(\tau-4)$, which gives $x(t) = U(t-4)$.

Thus, $y(t)$ is the ouput of the system when the input is $U(t-4)$. Since the system is TI, $y(t+4)$ is the output of the system when the input is $U(t)$. So $y(t+4)$ is the unit step responce of the system. Since the system is Linear, thus

$$h(t) = \frac{dy(t+4)}{dt}, \quad t \ge 0.$$

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Another way to answer this question is the following:

$$y(t) = \int_4^{\infty} h(t-\tau)\,U(t-\tau)\,d\tau$$

$$y(t+4) = \int_4^{\infty} h(t+4-\tau)\,U(t+4-\tau)\,d\tau$$

$$= \int_{-\infty}^{t} h(\tau')\,U(\tau')\,d\tau'$$

Thus,

$$\frac{dy(t+4)}{dt} = h(t)\,U(t)$$

So, the statement is true.

4. i) $x_1(t) = e^{-t}U(t)$,

$$X_1(s) = \frac{1}{s+1},$$

$$y_1(t) = e^{-t}(1-t)U(t),$$

$$Y_1(s) = \frac{1}{s+1} - \frac{1}{(s+1)^2} = \frac{s}{(s+1)^2}$$

$$H(s) = \frac{Y_1(s)}{X_1(s)} = \frac{s}{s+1}$$

Thus, $h(t) = L_s^{-1}\left[\frac{s}{s+1}\right]$

$$= L_s^{-1}\left[1 - \frac{1}{s+1}\right]$$

$$= \delta(t) - e^{-t}U(t).$$

ii) $y_2(t) = \cos t\, U(t)$

$$Y_2(s) = L_s[\cos t\, U(t)]$$

$$= \frac{s}{s^2+1}.$$

$$X_2(s) = \frac{Y_2(s)}{H(s)} = \frac{\frac{s}{s^2+1}}{\frac{s}{s+1}} = \frac{s+1}{s^2+1}$$

$$x_2(t) = L_s^{-1}\left[\frac{s+1}{s^2+1}\right] = (\cos t + \sin t)\,U(t).$$

5. i) $Y(s) = \frac{s-4}{s^2+5s+4}$

$$= \frac{s-4}{(s+1)(s+4)}$$

We can let

$$H(s) = \frac{1}{s+1}, \quad X(s) = \frac{s-4}{s+4}.$$

$$h(t) = L_s^{-1}[H(s)] = e^{-t}U(t)$$

$$x(t) = L_s^{-1}[X(s)] = L_s^{-1}\left[\frac{s-4}{s+4}\right]$$

$$= L_s^{-1}\left[1 - \frac{8}{s+4}\right]$$

$$= \delta(t) - 8e^{-4t}U(t)$$

Another way of splitting $Y(s)$ is

$$H(s) = \frac{1}{s+4}, \quad X(s) = \frac{s-4}{s+1}$$

$$h(t) = L_s^{-1}[H(s)] = e^{-4t}U(t),$$

$$x(t) = L_s^{-1}[X(s)] = L_s^{-1}\left[\frac{s-4}{s+1}\right]$$

$$= L_s^{-1}\left[1 - \frac{5}{s+1}\right]$$

$$= \delta(t) - 5e^{-t}U(t)$$

ii) $L_s\left[\int_0^t e^{-2\tau}\tau f(\tau)\,d\tau\right]$

$$= \frac{1}{s} L_s[e^{-2t} t f(t)]$$

$$= \frac{1}{s}\cdot\left\{L_s[t f(t)]\Big|_{s=s+2}\right\}$$

$$= \frac{1}{s}\cdot\left\{-\frac{d}{ds}F(s)\Big|_{s=s+2}\right\}$$

$$= \frac{1}{s}\cdot\left\{-\frac{d}{ds}F(s+2)\right\}$$

6. i) $F(s) = L_s[(\cos t + \sin t - e^{-t})U(t)]$

$$= \frac{s}{s^2+1} + \frac{1}{s^2+1} - \frac{1}{s+1}$$

$$= \frac{2s}{(s+1)(s^2+1)}$$

Let $H(s) = \frac{1}{s+1}$,

$$X(s) = \frac{2s}{s^2+1}.$$

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$h(t) = L_s^{-1}\left[\frac{1}{s+1}\right] = e^{-t}U(t)$

$x(t) = L_s^{-1}\left[\frac{2s}{s^2+1}\right] = 2\cos t\, U(t)$

Then, $f(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$

$= \int_{-\infty}^{\infty} e^{-(t-\tau)}U(t-\tau)\cdot 2\cos\tau U(\tau)$

$= \int_{-\infty}^{\infty} 2e^{-(t-\tau)}U(t-\tau)\cos\tau U(\tau)$

ii) $X(s) = L_s[\sin t\, U(t)] = \frac{1}{s^2+1}$

$Y(s) = L_s[t\cos t\, U(t)]$

$= -\frac{d}{ds}\frac{s}{s^2+1}$

$= \frac{s^2-1}{(s^2+1)^2}$.

$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{s^2-1}{(s^2+1)^2}}{\frac{1}{s^2+1}} = \frac{s^2-1}{s^2+1}$.

Thus, $\frac{d^2y(t)}{dt^2} + y(t) = \frac{d^2x(t)}{dt^2} - x(t)$.

$X'(s) = L_s[\sinh t\, U(t)]$

$= \frac{1}{s^2-1}$.

$Y'(s) = \frac{1}{s^2-1}\cdot\frac{s^2-1}{s^2+1}$

$= \frac{1}{s^2+1}$

$y'(t) = L_s^{-1}\left[\frac{1}{s^2+1}\right]$

$= \sin t\, U(t)$.

7. i) $s^2Y(s) + 3sY(s) + 2Y(s) = X(s)$

$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+3s+2}$

$h(t) = L_s^{-1}[H(s)] = L_s^{-1}\left[\frac{1}{s^2+3s+2}\right]$

$= L_s^{-1}\left[\frac{1}{(s+1)(s+2)}\right]$

$= L_s^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$

$= e^{-t}U(t) - e^{-2t}U(t)$

ii) Yes.

$H_1(s) = \frac{1}{s+1}$,

$H_2(s) = \frac{1}{s+2}$,

$h_1(t) = e^{-t}U(t)$,

$h_2(t) = e^{-2t}U(t)$

8. i) $\frac{(s+1)^2}{[(s+1)^2+1]^2} = \frac{s+1}{(s+1)^2+1}\cdot\frac{s+1}{(s+1)^2+1}$

$L_s^{-1}\left[\frac{s+1}{(s+1)^2+1}\right] = e^{-t}\cdot L_s^{-1}\left[\frac{s}{s^2+1}\right]$

$= e^{-t}\cos t\, U(t)$

Then, $L_s^{-1}\left[\frac{(s+1)^2}{[(s+1)^2+1]^2}\right]$

$= L_s^{-1}\left[\frac{s+1}{(s+1)^2+1}\right] * L_s^{-1}\left[\frac{s+1}{(s+1)^2+1}\right]$

$= \int_{-\infty}^{\infty} e^{-(t-\tau)}\cos(t-\tau)U(t-\tau)\cdot e^{-\tau}\cos\tau\, U(\tau)\, d\tau$

$= \int_{-\infty}^{\infty} e^{-t}\cos(t-\tau)\cos(\tau)U(t-\tau)\cdot U(\tau)d\tau$

$= \int_0^t e^{-t}\cos(t-\tau)\cos\tau\, d\tau$

ii) $H(s) = L_s[e^{-t}\cos t\, U(t)]$

$= L_s[\cos t\, U(t)]\big|_{s=s+1}$

$= \frac{s}{s^2+1}\Big|_{s=s+1}$

$= \frac{s+1}{(s+1)^2+1}$

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$$X(s) = L_s[e^{-t}\sin t\, U(t)]$$
$$= L_s[\sin t\, U(t)]\Big|_{s=s+1}$$
$$= \frac{1}{(s+1)^2+1}$$

Thus, $Y(s) = H(s)X(s)$
$$= \frac{s+1}{[(s+1)^2+1]^2}.$$

9. $X(s) = L_s[\delta(t) - 2e^{-2t}U(t)]$
$$= 1 - \frac{2}{s+2}$$
$$= \frac{s}{s+2}.$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\dfrac{2s}{s^3+s^2-4s-4}}{\dfrac{s}{s+2}}$$
$$= \frac{\dfrac{2s}{(s+1)(s+2)(s-2)}}{\dfrac{s}{s+2}}$$
$$= \frac{2}{(s+1)(s-2)}$$
$$= \frac{-\frac{2}{3}}{s+1} + \frac{\frac{2}{3}}{s-2}$$

$$h(t) = L_s^{-1}[H(s)] = \left(-\tfrac{2}{3}e^{-t} + \tfrac{2}{3}e^{2t}\right)U(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty}\left[-\tfrac{2}{3}e^{-(t-\tau)} + \tfrac{2}{3}e^{2(t-\tau)}\right]U(t-\tau)$$
$$\cdot\, x(\tau)\, d\tau$$