Systems and Signals

EE102

Lee, Spring 2009-10

Final Practice Solutions

Problem 1. Fourier Series

A linear amplifier has an output y that is proportional to the input x,

$$y = a_1 x$$

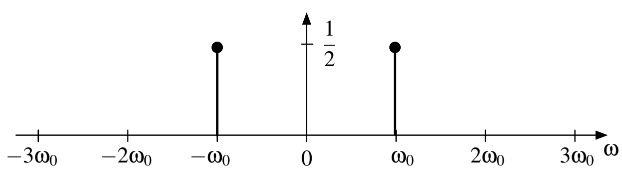
where a_1 is a constant. In practice an amplifier will have a more complex characteristic

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

If we apply an input

$$x(t) = \cos(\omega_0 t)$$

ideally we would get an output spectrum that looks like



where we've assumed $a_1 = 1$ for simplicity.

In practice we get something different, and this tells us something about the amplifier characteristic.

For each of the following amplifier characteristics, determine what the output Fourier series spectrum looks like when the inputs is $x(t) = \cos(\omega_0 t)$.

Hint: What does the spectra of $\cos^n(t)$ look like for different n? Don't integrate!

Solution: We can find the Fourier transform of $\cos^2(\omega_0 t)$ easily by frequency domain convolution

$$\begin{aligned}\mathcal{F}\left[\cos^2(\omega_0 t)\right] &= \frac{1}{2\pi}\left(\pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)) * (\pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))\right) \\ &= \frac{\pi}{2}\left(\delta(\omega-2\omega_0)+2\delta(\omega)+\delta(\omega+2\omega_0)\right)\end{aligned}$$

The Fourier series coefficients are just the Fourier transform coefficients divided by 2π, so

$$D_{\pm 2} = \frac{1}{4}, \text{ and } D_0 = \frac{1}{2}.$$

and all of the other coefficients are zero. Similarly, you found $\cos^3(\omega_0 t)$ to be

$$\begin{aligned}
\mathcal{F}\left[\cos^3(\omega_0 t)\right] &= \frac{1}{2\pi}\mathcal{F}\left[\cos^2(\omega_0 t)\right] * \mathcal{F}\left[\cos(\omega_0 t)\right] \\
&= \frac{1}{2\pi}\left(\frac{\pi}{2}\left(\delta(\omega - 2\omega_0) + 2\delta(\omega) + \delta(\omega + 2\omega_0)\right) * \left(\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))\right)\right) \\
&= \frac{\pi}{4}\left(\delta(\omega - 3\omega_0) + 3\delta(\omega - \omega_0) + 3\delta(\omega + \omega_0) + \delta(\omega + 3\omega_0)\right)
\end{aligned}$$

The Fourier series coefficients are then

$$D_{\pm 3} = \frac{1}{8}, \text{ and } D_{\pm 1} = \frac{3}{8}$$

a) Find the output Fourier series spectrum when the amplifier characteristic is

$$y = x + 0.1\ x^3.$$

Solution:

$$y(t) = \cos(\omega_0 t) + 0.1 \cos^3(\omega_0 t)$$

The Fourier series spectrum of $y(t)$ will be the sum of the spectrum for $\cos(\omega_0 t)$ and the spectrum $0.1\cos^3(\omega_0 t)$. The Fourier series spectrum of $\cos(\omega_0 t)$ is

$$D_{\pm 1} = \frac{1}{2}$$

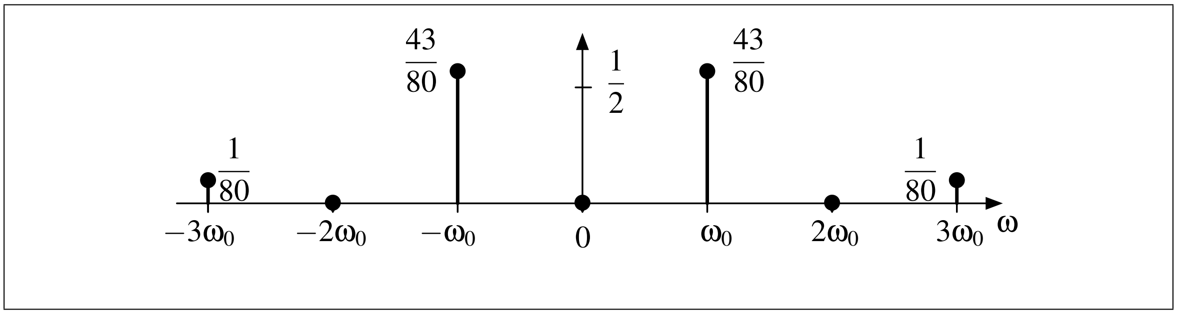
The Fourier series spectrum of $0.1\cos^3(\omega_0 t)$ is

$$\begin{aligned}
D_{\pm 3} &= (0.1)\frac{1}{8} = \frac{1}{80} \\
D_{\pm 1} &= (0.1)\frac{3}{8} = \frac{3}{80}
\end{aligned}$$

The sum of these is then

$$\begin{aligned}
D_{\pm 3} &= \frac{1}{80} \\
D_{\pm 1} &= \frac{1}{2} + \frac{3}{80} = \frac{43}{80}
\end{aligned}$$

This is plotted below.



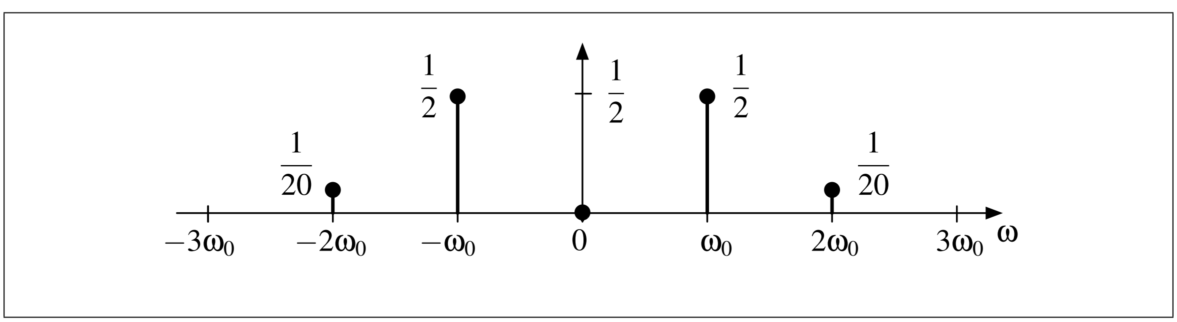
b) Find the output Fourier series spectrum when the amplifier characteristic is

$$y = -0.1 + x + 0.2\ x^2$$

Solution: Again, the Fourier series coefficients are the sum of the coefficients of each term. These are

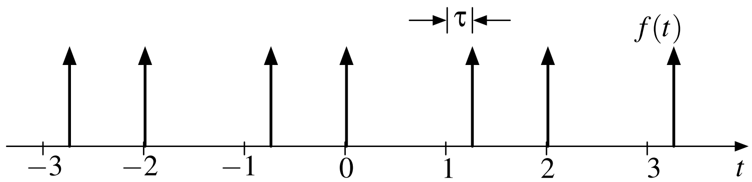
$$\begin{aligned} D_{\pm 2} &= 0.2\left(\frac{1}{4}\right) = \frac{1}{20} \\ D_{\pm 1} &= \frac{1}{2} \\ D_0 &= -0.1 + (0.2)\left(\frac{1}{2}\right) = 0 \end{aligned}$$

This is plotted below.



Problem 2. Sampling Timing Errors

Imperfections in a sampler cause characteristic artifacts in the sampled signal. In this problem we will look at the case where the sample timing is non-uniform, as shown below



The sampling function $f(t)$ has its odd samples delayed by a small time τ.

a) Write an expression for $f(t)$ in terms of two uniformly spaced sampling functions.

Solution: The even samples are a δ train separated by 2, with no shift. The odd samples are also a δ train separated by 2, but delayed by $1+\tau$. Adding these together

$$f(t) = \delta_2(t) + \delta_2(t - (1 + \tau))$$

$f(t) = \delta_2(t) + \delta_2(t - (1 + \tau))$

b) Find $F(j\omega)$, the Fourier transform of $f(t)$. Express the impulse trains as sums, and simplify.

Solution: The Fourier transform is

$$\begin{aligned} F(j\omega) &= \pi\delta_\pi(\omega) + \pi\delta_\pi(\omega)e^{-j\omega(1+\tau)} \\ &= \pi\delta_\pi(\omega)(1 + e^{-j\omega(1+\tau)}) \end{aligned}$$

where $\omega_0 = \frac{2\pi}{2} = \pi$ for the impulse train.

Expanding the impulse train,

$$\begin{aligned} F(j\omega) &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + e^{-j\omega(1+\tau)}) \\ &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + e^{-jn\pi(1+\tau)}) \\ &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + e^{-jn\pi}e^{-jn\pi\tau}) \\ &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + (-1)^n e^{-jn\pi\tau}) \end{aligned}$$

$$F(j\omega) = \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + (-1)^n e^{-jn\pi\tau})$$

c) Find $F(j\omega)$ for the case where $\tau = 0$, and show that this is what you expect.

Solution: If $\tau = 0$

$$\begin{aligned} F(j\omega) &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + (-1)^n e^{-jn\pi\tau}) \\ &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + (-1)^n) \end{aligned}$$

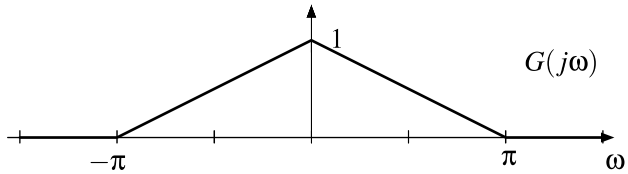
The $(1+(-1)^n)$ term is 2 for n even, and 0 for n odd. Hence, the odd terms drop, and we get a factor of two for the remaining even terms,

$$\begin{aligned} F(j\omega) &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - 2n\pi)(2) \\ &= 2\pi \sum_{n=-\infty}^{\infty} \delta(\omega - 2n\pi) \\ &= 2\pi\delta_{2\pi}(\omega) \end{aligned}$$

which is the Fourier transform of $\delta_1(t)$. This is what we expect. As τ goes to zero, we expect that the non-uniform sampling case should go to the uniform case. This will provide a reality check for the next part.

When $\tau = 0$, then $F(j\omega) = 2\pi\delta_{2\pi}(\omega)$

d) Assume the signal we are sampling has a Fourier transform



Sketch the Fourier transform of the sampled signal. Include the baseband replica, and the replicas at $\omega = \pm\pi$. Assume that τ is small, so that $e^{j\omega\tau} \simeq 1 + j\omega\tau$.

Solution: The sampled signal is $f(t)g(t)$, which has a Fourier transform

$$\begin{aligned}
\bar{G}(j\omega) &= \frac{1}{2\pi}\left[F(j\omega) * G(j\omega)\right] \\
&= \frac{1}{2\pi}\left[\left[\pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + (-1)^n e^{-jn\pi\tau})\right] * \Delta(\omega/\pi)\right] \\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty} \Delta\left(\frac{\omega - n\pi}{\pi}\right)(1 + (-1)^n e^{-jn\pi\tau})
\end{aligned}$$

We are interested in the baseband replica (n=0) , and the replicas at $\pm\pi$ (n= $\pm$ 1). For $n = 0$,

$$\bar{G}_0(j\omega) = \frac{1}{2}\Delta\left(\frac{\omega}{\pi}\right)(1+1) = \Delta\left(\frac{\omega}{\pi}\right)$$

which is the same as $G(j\omega)$. For $n = 1$,

$$\begin{aligned}
\bar{G}_1(j\omega) &= \frac{1}{2}\Delta\left(\frac{\omega - \pi}{\pi}\right)(1 + (-1)^1 e^{-j\pi\tau}) \\
&= \frac{1}{2}\Delta\left(\frac{\omega - \pi}{\pi}\right)(1 - e^{-j\pi\tau})
\end{aligned}$$

If we approximate $e^{-j\pi\tau} \simeq 1 - j\pi\tau$,

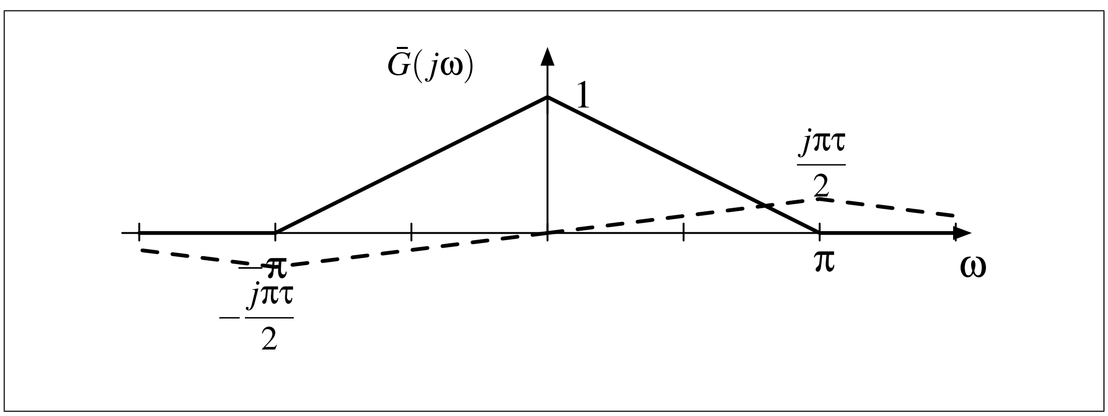
$$\begin{aligned}
\bar{G}_1(j\omega) &= \frac{1}{2}\Delta\left(\frac{\omega - \pi}{\pi}\right)(1 - (1 - j\pi\tau)) \\
&= \frac{j\pi\tau}{2}\Delta\left(\frac{\omega - \pi}{\pi}\right)
\end{aligned}$$

This is a replica of $G(j\omega)$ centered at $\omega = \pi$, multiplied by $j\pi\tau/2$. It is imaginary, and proportional to τ, so that as τ goes to zero, this replica disappears as we'd expect.

For $n = -1$ we get the same type of term, but with the negative sign,

$$\bar{G}_{-1}(j\omega) = \frac{-j\pi\tau}{2}\Delta\left(\frac{\omega+\pi}{\pi}\right)$$

This is a replica of $G(j\omega)$ centered at $\omega = -\pi$, and scaled by $-j\pi\tau/2$. If we sketch these three terms, the result is as shown below.



e) If we know $g(t)$ is real and even, can we recover $g(t)$ from the non-uniform samples $g(t)f(t)$?

Solution We know that in the limit at τ goes to zero, that we can perfectly reconstruct $g(t)$, since we will then be sampling at the Nyquist rate. From the answer to the previous part, this does indeed happen. The replicas at $\pm\pi$ are proportional to τ, and will go to zero.

Looking at the solution for the previous part, we can see that the part of the spectrum we want is all in the real component. If $g(t)$ is real and even, then $G(j\omega)$ is real and even. Hence, if we lowpass filter, and take the real part of the spectrum, we can recover $G(j\omega)$.

It actually turns out that if we know τ, we can recover any bandlimited $g(t)$. The symmetry is not required. This is a harder problem, but you can figure it out if you are interested.

Can we recover $g(t)$?	[yes]	no

Problem 3. Laplace Transforms and Stability

For each of these assertions, determine whether they are true or false. Provide an argument for your conclusion. Remember that an unstable system has poles either in the right-half plane (diverging solutions) or on the $j\omega$ axis (oscillating or constant solutions).

a) Let $h(t)$ be the impulse response of a stable, causal system. Then $\frac{d}{dt}h(t)$ is also stable.

Solution:

$$\mathcal{L}\left[\frac{d}{dt}h(t)\right] = sH(s)$$

Differentiating in time adds a *zero* at $s = 0$ to the Laplace transform. This doesn't effect stability, so a stable system will remain stable. The assertion is then true.

The assertion is (circle one):	[true]	false

b) Let $h(t)$ be the impulse response of a stable, causal system. Then $\int_{-\infty}^{t} h(\tau)d\tau$ must be unstable.

Solution:

$$\mathcal{L}\left[\int_{-\infty}^{t} h(\tau)d\tau\right] = \frac{1}{s}H(s)$$

This adds a pole at $s = 0$, which would seem to result in an unstable system by the definition given above. However, $H(s)$ may have a zero at $s = 0$, in which case the two cancel, and the system remains stable. Hence, the result is not necessarily an unstable system.

The assertion is (circle one):	true	[false]

c) $H(s)$ is the transfer function of a stable, causal system. The zeros of $H(s)$ must be in the right-half plane for the inverse system $H_{inv}(s)$ to be stable.

Solution:

The transfer function of an inverse system is

$$H_{inv}(s) = \frac{1}{H(s)}$$

Hence, the zeros of $H(s)$ become the poles of $H_{inv}(s)$ (and the poles of $H(s)$ become the zeros of $H_{inv}(s)$, but that doesn't concern us here).

If $H(s)$ has *zeros* in the right-half plane, $H_{inv}(s)$ will have *poles* is the right-half plane. These correspond to increasing exponential solutions, and are unstable. The assertion is then false.

The assertion is (circle one):	true	false