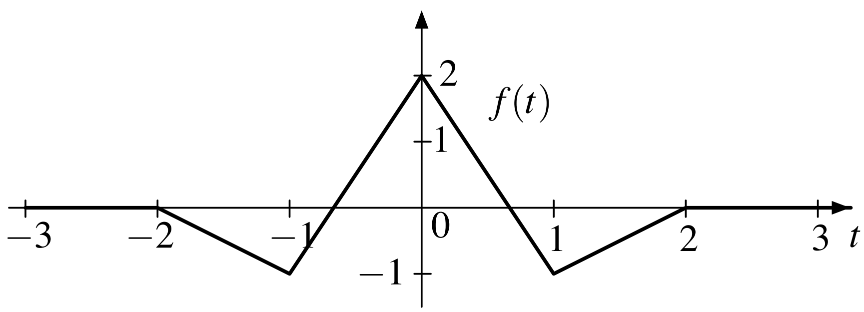
Midterm Practice Problems Solutions

Problem 1. Computing Fourier Transforms

The signal $f(t)$ is plotted below:



a) Find an expression for $f(t)$ in terms sums and convolutions of signals defined for all t such as $\mathrm{rect}(t)$ and $\Delta(t)$ (*i.e.* do *not* express it as a piecewise signal, defined over intervals).

Solution:
There are several solutions. Two simple examples are

$$f(t) = 2\Delta(t) - \Delta(t-1) - \Delta(t+1)$$

and

$$f(t) = 4\Delta(t) - 2\Delta(t/2)$$

b) Find the Fourier transform of $f(t)$. Simplify your answer.

Solution:

The Fourier transform of the first solution is

$$F(j\omega) = 2\mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right) - e^{j\omega}\mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right) - e^{-j\omega}\mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right)$$

which can be simplified to

$$F(j\omega) = 2\mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right)(1-\cos(\omega)).$$

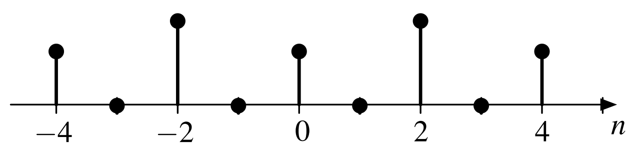
The second solution has the Fourier transform

$$F(j\omega) = 4\mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right) - 4\mathrm{sinc}^2\left(\frac{\omega}{\pi}\right).$$

Problem 2. Fourier Series

a) You compute the Fourier series of a signal $f(t)$ using a period T_0. You find that all of the odd Fourier series coefficients are zero (i.e. $D_1, D_{-1}, D_3, D_{-3}, \cdots$ are all zero). What can you conclude about $f(t)$?

Solution
if we sketch what the spectrum looks like, we get



The frequencies are all multiples of $2\omega_0$. This is a Fourier series that has a fundamental frequency of $2\omega_0$. The period of a signal with this fundmental frequency is

$$T_0' = \frac{2\pi}{2\omega_0} = \frac{1}{2}\left(\frac{2\pi}{\omega_0}\right) = \frac{1}{2}T_0.$$

Hence $f(t)$ has a period $T_0/2$. What we have computed is the Fourier series over an interval that includes two cycles of $f(t)$.

Another acceptable answer is that $f(t) = f(t \pm T_0/2)$, since this means the same thing. $f(t)$ is: periodic, with period $T_0/2$.

b) Two functions $f(t)$ and $\tilde{f}(t)$ have Fourier series coefficients D_n and $\tilde{D}_n$, respectively, and

$$\begin{aligned} \left|\tilde{D}_n\right| &= |D_n| \\ \angle\tilde{D}_n &= \angle D_n + n\pi \end{aligned}$$

Find a simple expression for $\tilde{f}(t)$ in terms of $f(t)$.

Solution:
The signal $\tilde{f}(t)$ is

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} \tilde{D}_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \left(e^{j\pi n} D_n\right) e^{jn\omega_0 t}$$

We know from the homework (HW3 Q9), that a delayed signal has the same Fourier series coefficients multiplied by a complex phase factor.

$$f(t-\tau) = \sum_{n=-\infty}^{\infty} \left(e^{-j\omega_0 \tau n} D_n\right) e^{jn\omega_0 t}$$

This is the same form. To figure out what τ is we need to equate the coefficients of the exponentials, so

$$-\omega_0 \tau = \pi$$

and

$$\tau = -\frac{\pi}{\omega_0} = -\frac{1}{2}\left(\frac{2\pi}{\omega_0}\right) = -\frac{1}{2}T_0.$$

Hence

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} \left(e^{j\omega_0(-\frac{1}{2}T_0)n} D_n\right) e^{jn\omega_0 t} = f(t + T_0/2).$$

This also equals $f(t - T_0/2)$ since $f(t)$ has a period T_0. $\tilde{f}(t) = f(t + T_0/2)$.