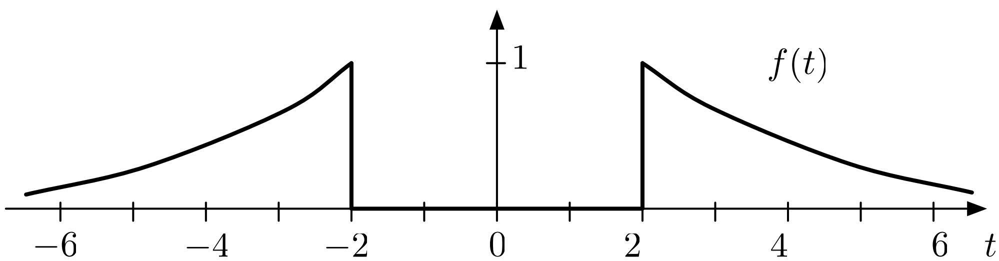
Systems and Signals — Lee, Spring 2009-10
EE102

# Midterm Exam Solutions

*Problem 1. Computing Fourier Transforms* *(30 points)*

The signal $f(t)$ is plotted below:




It consists of the signal $u(t)e^{-t/2}$ that has been shifted, and a shifted and reversed version of the same signal.

a) Find an expression for $f(t)$

**Solution:**
The signal is $u(t)e^{-t/2}$ delayed by 2, which is

$$u(t-2)e^{-(t-2)/2}$$

and a time reversed version of the same signal

$$u((-t)-2)e^{-((-t)-2)/2} = u(-(t+2))e^{(t+2)/2}$$

The result is

$$f(t) = u(t-2)e^{-(t-2)/2} + u(-(t+2))e^{(t+2)/2}$$

$f(t) = u(t-2)e^{-(t-2)/2} + u(-(t+2))e^{(t+2)/2}$

b) Find $F(j\omega)$, the Fourier transform of $f(t)$. Make sure to simplify your answer.

**Solution**

From the Lecture notes we know that

$$\mathcal{F}\left[u(t)e^{-at}\right] = \frac{1}{a + j\omega}$$

and

$$\mathcal{F}\left[u(-t)e^{at}\right] = \frac{1}{a - j\omega}$$

Using these, plus the shift theorem, and simplifying

$$\begin{aligned}
\mathcal{F}\left[u(t-2)e^{-(t-2)/2} + u(-(t+2))e^{(t+2)/2}\right] &= \frac{e^{-j2\omega}}{\frac{1}{2} + j\omega} + \frac{e^{j2\omega}}{\frac{1}{2} - j\omega} \\
&= \frac{e^{-j2\omega}\left(\frac{1}{2} - j\omega\right) + e^{j2\omega}\left(\frac{1}{2} + j\omega\right)}{\frac{1}{4} + \omega^2} \\
&= \frac{\frac{1}{2}(e^{-j2\omega} + e^{j2\omega}) + j\omega(e^{j2\omega} - e^{-2j\omega})}{\frac{1}{4} + \omega^2} \\
&= \frac{\cos(2\omega t) + j\omega(j2\sin(2\omega t))}{\frac{1}{4} + \omega^2} \\
&= \frac{\cos(2\omega t) - 2\omega\sin(2\omega t)}{\frac{1}{4} + \omega^2}
\end{aligned}$$

As a check, note that this is a real and even function of $\omega$, which is what we'd expect given that $f(t)$ is real and even.

$$F(j\omega) = \frac{\cos(2\omega t) - 2\omega\sin(2\omega t)}{\frac{1}{4} + \omega^2}$$

*Problem 2. Properties of convolution* *(20 Points)*

Determine whether the assertions are true or false, and provide a supporting argument.

a) The convolution of two odd signals is an odd signal.

**Solution:**
Let $f$ and $g$ be two odd signals, By the convolution theorem,

$$\mathcal{F}\left[(f * g)(t)\right] = F(j\omega)G(j\omega)$$

Since $f(t)$ is odd, then $F(j\omega)$ is odd. Similarly, $g(t)$ is odd, so $G(j\omega)$ is odd. The product of two odd function is even, so the Fourier transform of the convolution $F(j\omega)G(j\omega)$ is even. Since an even signal has an even transform, then $(f * g)(t)$ is even, and the assertion is false.

| | | |
|---|---|---|
| The assertion is (circle one) | true | <u>false</u> |

b) If $(f * g)(t) = h(t)$, and $f_d(t) = f(t - a)$, and $g_d(t) = g(t - b)$, then

$$(f_d * g_d)(t) = h(t - b - a).$$

**Solution:**
There are several ways to approach this. An easy one is to use the convolution theorem. First note that $H(j\omega) = F(j\omega)G(j\omega)$. Then consider

$$\begin{aligned}
\mathcal{F}\left[(f_d * g_d)(t)\right] &= \mathcal{F}\left[f_d(t)\right]\mathcal{F}\left[g_d(t)\right] \\
&= e^{-j\omega a}F(j\omega)e^{-j\omega b}G(j\omega) \\
&= e^{-j\omega(a+b)}F(j\omega)G(j\omega) \\
&= e^{-j\omega(a+b)}H(j\omega)
\end{aligned}$$

This is the Fourier transform of $h(t - (a + b))$, so the assertion is true.

| | | |
|---|---|---|
| The assertion is (circle one) | <u>true</u> | false |

*Problem 3. Parseval's Theorem (30points).*

The spectrum

$$Y(j\omega) = \frac{1}{1 - j\omega}$$

has an energy

$$E_Y = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{1}{1 - j\omega} \right|^2 d\omega.$$

Find $E_Y$.

***Solution***

By Parseval's theorem

$$E_Y = \int_{-\infty}^{\infty} |y(t)|^2 \ dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(j\omega)|^2 \, d\omega$$

We identify

$$Y(j\omega) = \frac{1}{1 - j\omega}$$

which has an inverse Fourier transform

$$y(t) = u(-t)e^t$$

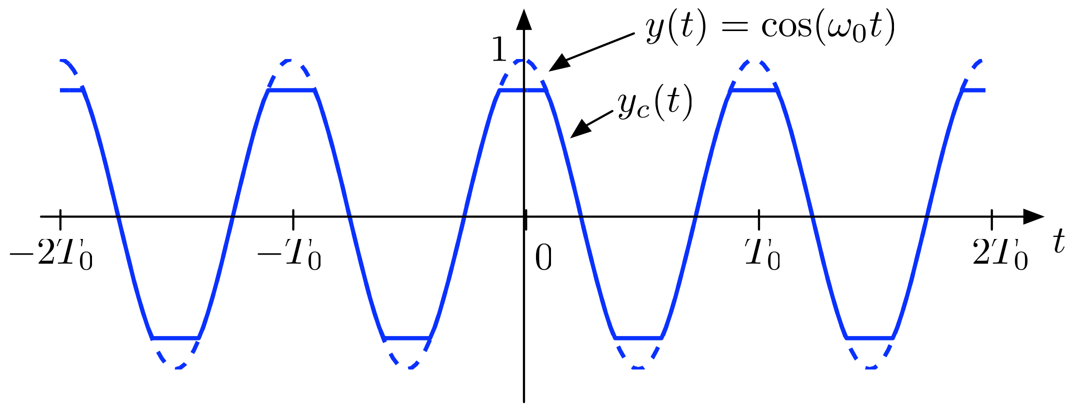
Note that this is a decreasing exponential that goes to the left. Then

$$\begin{aligned} E_Y &= \int_{-\infty}^{\infty} |y(t)|^2 \ dt \\ &= \int_{-\infty}^{\infty} \left| u(-t)e^t \right|^2 \ dt \\ &= \int_{-\infty}^{0} e^{2t} \ dt \\ &= \frac{1}{2} \ e^{2t} \Big|_{-\infty}^{0} \\ &= \frac{1}{2}(1 - 0) \\ &= \frac{1}{2} \end{aligned}$$

$$E_Y = \frac{1}{2}$$

*Problem 4. Fourier Series (25 points).*

An amplifier has an input of $x(t) = \cos(\omega_0 t)$. Unfortunately, the output is limited by the rail voltage of the amplifier, and the result is the clipped waveform $y_c(t)$ shown below:




We will use the Fourier series to characterize the effect of this clipping.

a) Is the amplifier a linear system? Is it time-invariant?

**Solution:**
The amplifier clips at the same voltage, for any input. Hence if we scale the input, the cosine will be clipped at a different level, and won't be a scaled version of the same clipped waveform. Hence, this is a *non-linear* system. However, we can still use frequency domain methods to analyze it!

The clipping depends on the signal itself. If we delay the signal, it will still be clipped at the same level, and will produce the a delayed version of the same clipped signal. Hence, this system is *time invariant*.

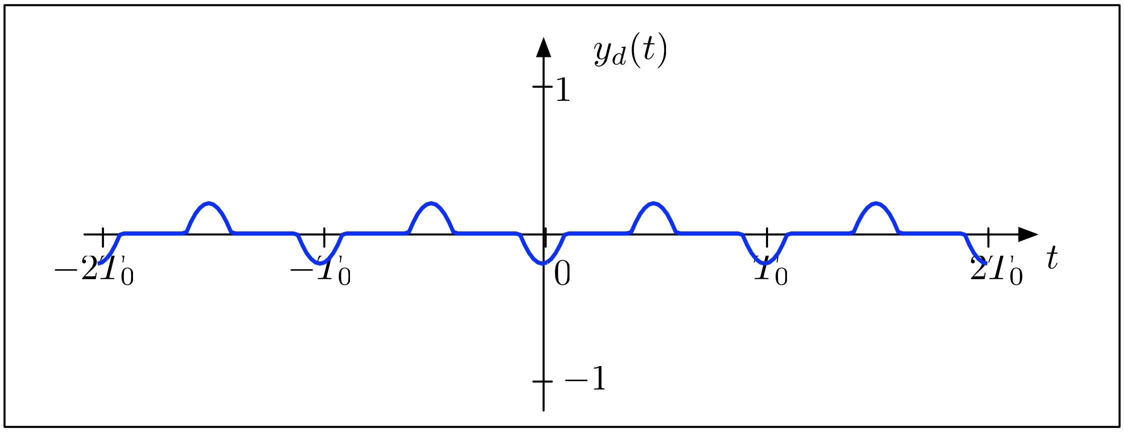
| Linear? | yes | no | Time Invariant? | yes | no |
|---|---|---|---|---|---|

b) We can think of the clipped signal as the ideal signal $y(t) = \cos(\omega_0 t)$ plus a distortion term $y_d(t)$, so that

$$y_c(t) = y(t) + y_d(t).$$

Sketch the error signal $y_d(t)$.

**Solution:**




c) What is the symmetry of the Fourier series coefficients of the error term $y_d(t)$? Are they real, imaginary, or complex?

**Solution:**
The signal is *real and even* so this means that the Fourier series coefficients are *real and even*.

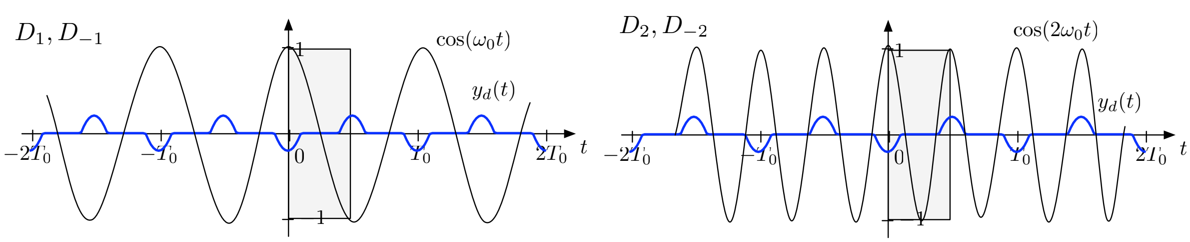
| | | |
|---|---|---|
| Symmetry: Even | Real, Imaginary, or Complex? | : |

d) Which of the Fourier series coefficients of $y_d(t)$ are non-zero? For example, is $D_0$ non-zero, or $D_1$ non-zero? You don't need to calculate the coefficients.
*Hint:* When considering the Fourier series coefficients integral over the whole period $-T_0/2$ to $T_0/2$, look for symmetries over *half* the period, such as $-T_0/2$ to 0, and 0 to $T_0/2$.

**Solution:**
First, since the Fourier series coefficients are real and even, the Fourier series involves only cosine terms, since cosine is real and even

$$y_d = \sum_{n=-\infty}^{\infty} D_n \cos(n\omega_0 t)$$

From the plot of part (b) we can see that $D_0 = 0$, since that is just the average value. For the higher order terms, the Fourier series coefficients will be $y_d(t)$ multiplied by $\cos(n\omega_0 t)$, integrated over one period. This is illustrated below for $n = \pm 1$ and $n = \pm 2$.




The shaded region is half of one period, from 0 to $T_0/2$. For $D_1, D_{-1}$, the integral over the shaded region is an odd function (centered on $T_0/4$) multiplied by an odd function $y_d(t)$, which is even. The integral will be non-zero, as will all of the $n$ odd terms.

For $D_2, D_{-2}$, the integral over the half period is of an even function times an odd function, which is odd. The integral will then be zero. This will be true for all of the $n$ even terms.

Hence the non-zero coefficients are $n$ odd.

This part was worth 10 points, and (a)-(c) were 5 points each. Few points were giving for correct answers without supporting arguments.

| |
|---|
| The non-zero coefficient are: $n$ odd |