Systems and Signals — Lee, Fall 2011-12
EE102

Midterm Exam 1

NAME:

You have 2 hours for 4 questions.

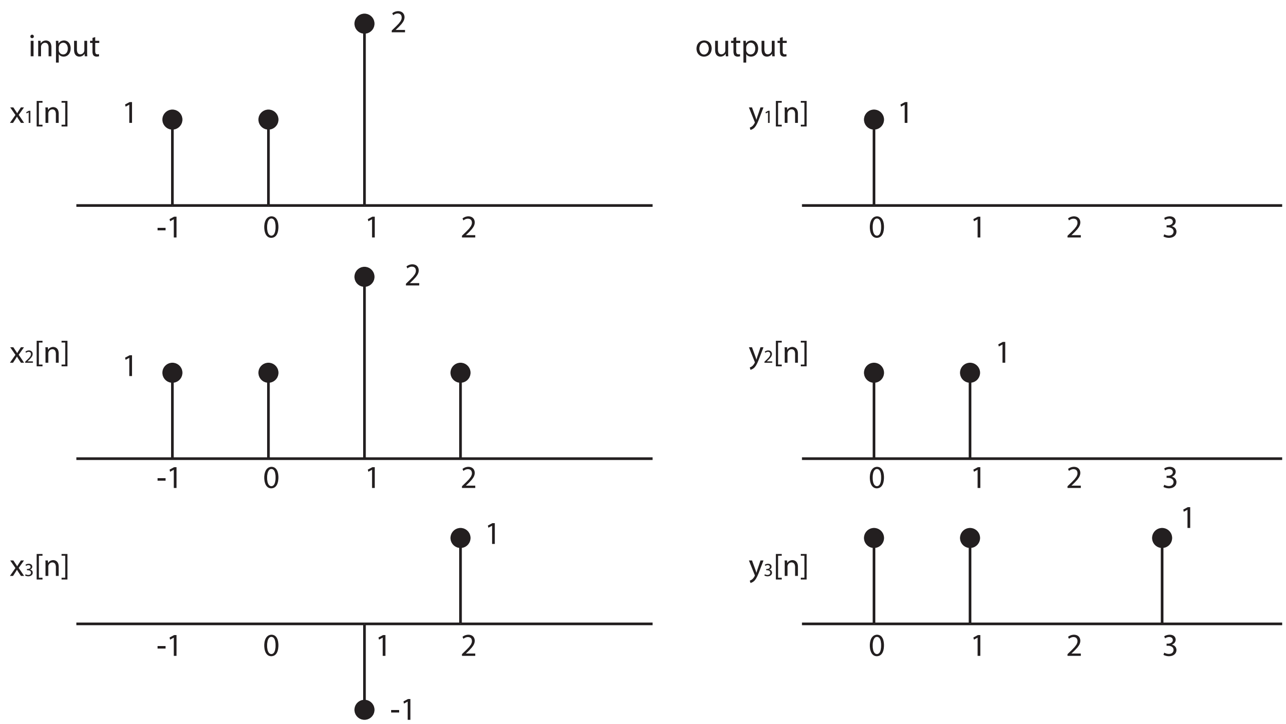
- Show enough (neat) work in the clear spaces on this exam to convince us that you derived, not guessed, your answers.
- Put your final answers in the boxes at the bottom of the page.

Closed notes, closed book, 1 letter sized sheet allowed.

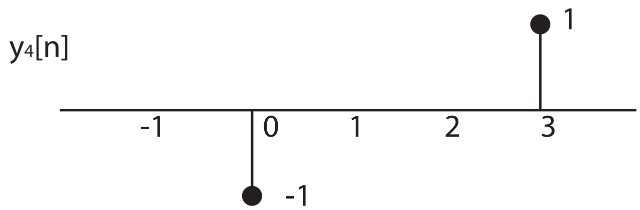
Problem	Score
1	
2	
3	
4	
Total	

Problem 1. *(25 Points)*

A LINEAR system is shown below. Suppose there are three test signals ($x_1[n]$, $x_2[n]$, $x_3[n]$), for which the system outputs are $y_1[n]$, $y_2[n]$, $y_3[n]$, respectively.

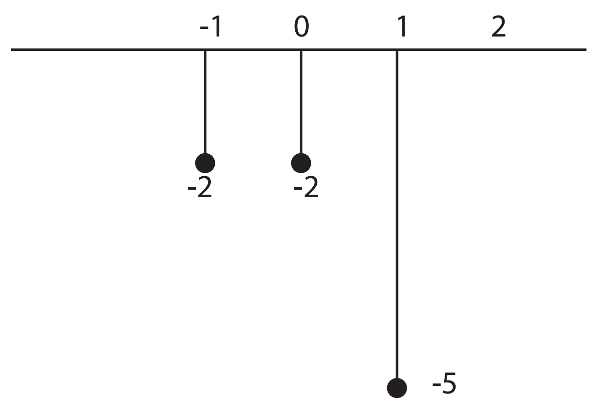


1. If the output of the system ($y_4[n]$) when $x_4[n]$ is the input as shown below, find the input signal $x_4[n]$ as a linear combination of $x_1[n]$, $x_2[n]$, and $x_3[n]$. Also, plot $x_4[n]$.



Solution:

$y_4[n] = -y_1[n] - y_2[n] + y_3[n]$. Therefore, $x_4[n] = -x_1[n] - x_2[n] + x_3[n]$.



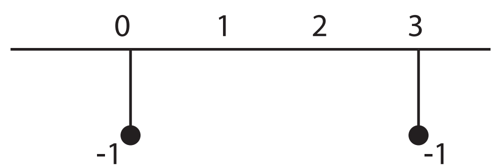
2. What are the outputs of the system when the inputs are $\delta[n-1]$ and $\delta[n-2]$, respectively? Also, plot the two outputs seperately.

Solution:

$$\delta[n-1] = x_2[n] - x_1[n] - x_3[n]$$

therefore, the output is

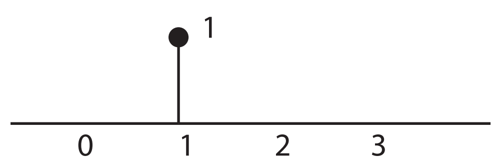
$$y_2[n] - y_1[n] - y_3[n]$$



$$\delta[n-2] = x_2[n] - x_1[n]$$

therefore, the output is

$$y_2[n] - y_1[n]$$



3. Based on what you have found from the last question, do you think this system is LTI? Why? Please explain your reason in detail.

   Solution:

   NO. Because when the input signal $\delta[n-1]$ gets delayed by 1 to $\delta[n-2]$, the output is not the output of input $\delta[n-1]$ delayed by 1.

Problem 2. *(25 Points)*

Determine the Linearity and Time Invariance of the following systems with the given input $x(t)$ and output $y(t)$ relationship.

1. $y(t) = (t-1)^2 x(t-4)$;

   Solution:

   Linearity:

   Suppose we have two signals $x_1(t)$ and $x_2(t)$ with corresponding outputs $y_1(t) = (t-1)^2 x_1(t-4)$ and $y_2(t) = (t-1)^2 x_2(t-4)$, respectively. If we input $x(t) = ax_1(t) + bx_2(t)$ into the system, the output is

$$\begin{aligned} y(t) &= (t-1)^2\{ax_1(t-4) + bx_2(t-4)\} \\ &= a(t-1)^2 x_1(t-4) + b(t-1)^2 x_2(t-4) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

   Therefore, the system is linear.

   Time Invariance:

   Suppose we have a delayed signal $x_d(t) = x(t-T)$. The corresponding output of the system is,

$$\begin{aligned} y_d(t) &= (t-1)^2 x_d(t-4) \\ &= (t-1)^2 x(t-4-T) \end{aligned}$$

   On the tneother hand, the delayed output is,

$$\begin{aligned} y(t-T) &= (t-T-1)^2 x(t-T-4) \\ &\neq y_d(t) \end{aligned}$$

   Therefore, the system is Time Variant.

The system is Linear	[True]	False
The system is Time Invariant	True	[False]

2. $y(t) = x(t)^2 + 6x(t-1) + 5x(t-2);$

Solution:

Linearity:

Suppose we have two signals $x_1(t)$ and $x_2(t)$ with corresponding outputs $y_1(t) = x_1(t)^2 + 6x_1(t-1) + 5x_1(t-2)$ and $y_2(t) = x_2(t)^2 + 6x_2(t-1) + 5x_2(t-2)$, respectively. If we input $x(t) = ax_1(t) + bx_2(t)$ into the system, the output is

$$\begin{aligned} y(t) &= (ax_1(t) + bx_2(t))^2 + 6(ax_1(t-1) + bx_2(t-1)) + 5(ax_1(t-2) + bx_2(t-2)) \\ &\neq a(x_1(t)^2 + 6x_1(t-1) + 5x_1(t-2)) + b(x_1(t)^2 + 6x_1(t-1) + 5x_1(t-2)) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Therefore, the system is non-linear.

Time Invariance:

Suppose we have a delayed signal $x_d(t) = x(t-T)$. The corresponding output of the system is,

$$\begin{aligned} y_d(t) &= x_d(t)^2 + 6x_d(t-1) + 5x_d(t-2) \\ &= x(t-T)^2 + 6x(t-T-1) + 5x(t-T-2) \end{aligned} \tag{1}$$

On the other hand, the delayed output is,

$$\begin{aligned} y(t-T) &= x(t-T)^2 + 6x(t-T-1) + 5x(t-T-2); \\ &= y_d(t) \end{aligned} \tag{2}$$

Therefore, the system is Time Invariant.

The system is Linear	True	[False]
The system is Time Invariant	[True]	False

3. $y(t) = real\{x(t-1)\}$;

Solution:

$y(t) = \frac{x(t-1)+x^*(t-1)}{2}$

Linearity:

Suppose we have two signal $x_1(t)$ and $x_2(t)$ with corresponding outputs $y_1(t) = \frac{x_1(t-1)+x_1^*(t-1)}{2}$ and $y_2(t) = \frac{x_2(t-1)+x_2^*(t-1)}{2}$, respectively. If we input $x(t) = ax_1(t) + bx_2(t)$ into the system, the output is

$$\begin{aligned} y(t) &= \frac{ax_1(t-1) + bx_2(t-1) + a^*x_1^*(t-1) + b^*x_2^*(t-1)}{2} \\ &\neq a\frac{x_1(t-1) + x_1^*(t-1)}{2} + b\frac{x_2(t-1) + x_2^*(t-1)}{2} \\ &= ay_1(t) + by_2(t) \end{aligned} \tag{3}$$

Therefore, the system is non-linear because the scaling factor is not necessary to be real.

Time Invariance:

Suppose we have a delayed signal $x_d(t) = x(t-T)$. The corresponding output of the system is,

$$\begin{aligned} y_d(t) &= \frac{x_d(t-1) + x_d^*(t-1)}{2} \\ &= \frac{x(t-1-T) + x_d^*(t-1-T)}{2} \\ &= \frac{x(t-T-1) + x^*(t-T-1)}{2} \\ &= y(t-T) \end{aligned}$$

Therefore, the system is Time Invariant.

The system is Linear	True	[False]
The system is Time Invariant	[True]	False

4. $y(t) = \int_{-t}^{t} x(\tau - 2)d\tau$;

Solution:

Linearity:

Suppose we have two signal $x_1(t)$ and $x_2(t)$ which have output $y_1(t) = \int_{-t}^{t} x_1(\tau - 2)d\tau$ and $y_2(t) = \int_{-t}^{t} x_2(\tau - 2)d\tau$ respectively. If we input $x(t) = ax_1(t) + bx_2(t)$ into the system, the output is

$$\begin{aligned} y(t) &= \int_{-t}^{t} ax_1(\tau - 2) + bx_2(\tau - 2)d\tau \\ &= a\int_{-t}^{t} x_1(\tau - 2)d\tau + b\int_{-t}^{t} x_2(\tau - 2)d\tau \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Thus the system is linear.

Time Invariance:

Suppose we have a delayed signal $x_d(t) = x(t - T)$. It is input to the sytem:

$$\begin{aligned} y_d(t) &= \int_{-t}^{t} x_d(\tau - 2)d\tau \\ &= \int_{-t}^{t} x(\tau - 2 - T)d\tau \\ &= \int_{-t-2-T}^{t-2-T} x(\tau)d\tau \end{aligned}$$

On another hand, the delayed output :

$$\begin{aligned} y(t - T) &= \int_{-t+T}^{t-T} x(\tau - 2)d\tau \\ &= \int_{-t+T-2}^{t-T-2} x(\tau)d\tau \\ &\neq y_d(t) \end{aligned}$$

Thus the system is Time Variant.

The system is Linear	**True**	False
The system is Time Invariant	True	**False**

Problem 3. *(25 Points)*

1. Let $x(t)$, $y(t)$, and $h(t)$ be the input, output, and impulse response functions of a system, respectively. If the input-output relationship of that system is described by the *convolution integral*, i.e.

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau,$$

prove that the system is *linear* and *time-invariant*.

Solution:

$$\begin{aligned} F[ax_1(t) + bx_2(t)] &= \int_{-\infty}^{\infty} h(t-\tau)[ax_1(\tau) + bx_2(\tau)]d\tau \\ &= a\int_{-\infty}^{\infty} h(t-\tau)x_1(\tau)d\tau + b\int_{-\infty}^{\infty} h(t-\tau)x_2(\tau)d\tau \\ &= aF[x_1(t)] + bF[x_2(t)] \end{aligned}$$

Therefore, this system is *linear*.

Delay the input:

$$\begin{aligned} F[x(t-T)] &= \int_{-\infty}^{\infty} h(t-\tau)x(\tau-T)d\tau, \quad \text{Let } \sigma = \tau - T \\ &= \int_{-\infty}^{\infty} h(t-\sigma-T)x(\sigma)d\sigma \end{aligned}$$

Delay the output:

$$y(t-T) = \int_{-\infty}^{\infty} h(t-T-\tau)x(\tau)d\tau$$

Therefore, this system is *time-invariant*.

2. If the above system is also *causal*, prove that

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)d\tau.$$

Solution: If the system is causal, $h(t) = h(t)u(t)$. Therefore, we have

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(t-\tau)u(t-\tau)x(\tau)d\tau \\ &= \int_{-\infty}^{t} h(t-\tau)x(\tau)d\tau \end{aligned}$$

3. Given a *linear* and *time-invariant* (LTI) system, if the input $x(t) = u(t-2)$ is applied to that system, we obtain $y(t) = \cosh(t-2)u(t-2)$ as the output. Determine the impulse response function $h(t)$ of that system. Then, using the $h(t)$ you have determined, write down the input-output relationship of the system.

   Reminder: $\cosh(t) = \frac{1}{2}[e^t + e^{-t}]$

   Solution: Since the system is *time-invariant*, the input $x(t) = u(t)$ results in the output $y(t) = \cosh(t)u(t)$.
   Moreover, if $\delta(t) = du(t)/dt$ is applied as an input, we obtain impulse response function $h(t)$ as the output. Therefore, we have

$$\begin{aligned} h(t) &= \frac{dy(t)}{dt} = [\cosh(t)u(t)]' = \cosh(t)'u(t) + \cosh(t)u(t)' \\ &= \sinh(t)u(t) + \cosh(t)\delta(t) = \sinh(t)u(t) + \cosh(0)\delta(t) \\ &= \sinh(t)u(t) + \delta(t) \end{aligned}$$

   Therefore,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} [\sinh(t-\tau)u(t-\tau) + \delta(t-\tau)]x(\tau)d\tau \\ &= x(t) + \int_{-\infty}^{\infty} [\sinh(t-\tau)u(t-\tau)]x(\tau)d\tau \end{aligned}$$

4. Is this system *BIBO stable*? Is it *causal*? Is it *memoryless*? Please explain your reason!

Solution: Because $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} |\sinh(t)u(t)|dt = \infty$, this system is *not BIBO stable*.

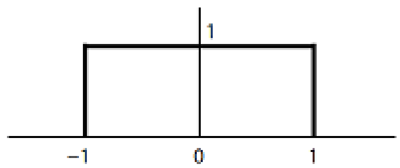
This system is *causal* because $h(t)u(t) = 0$, for $t < 0$.

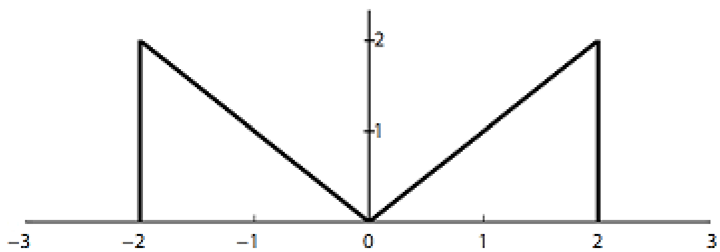
This is *not* a *memoryless* system because $y(t)$ depends on the past input from the integral.

The system is BIBO stable	True	[False]
The system is Causal	[True]	False
The system is Memoryless	True	[False]

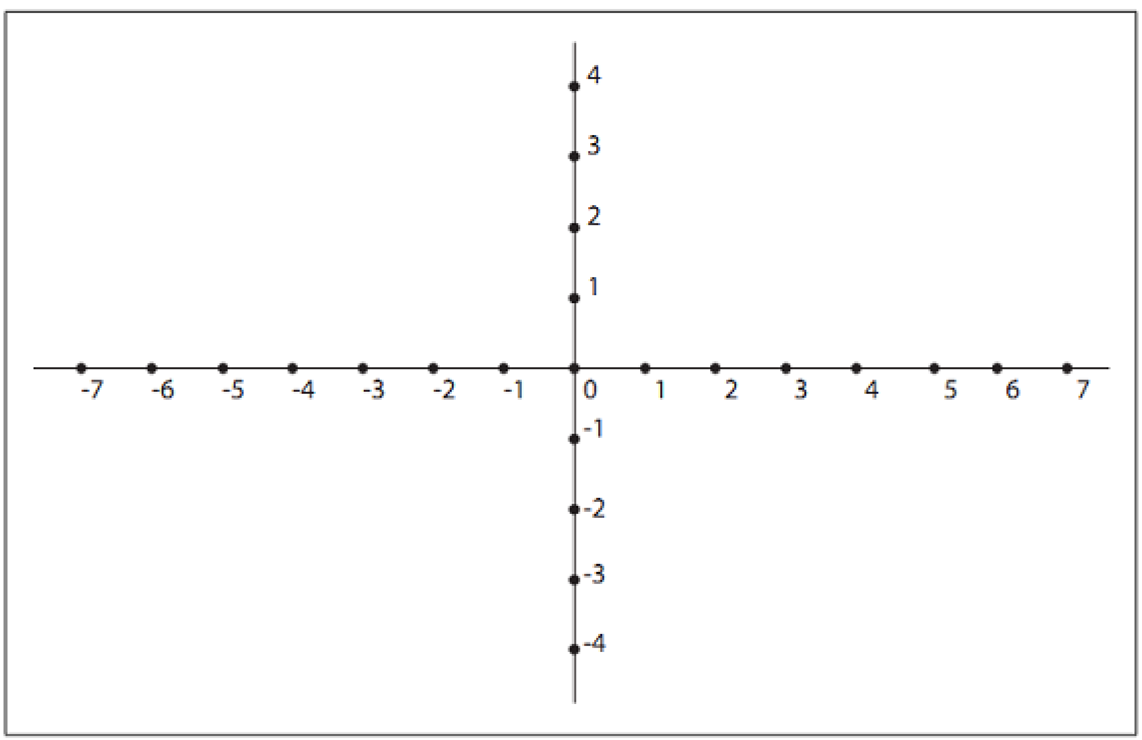
Problem 4. *(25 Points)*

Find the convolution of the two signals shown below **graphically**.





Plot your answer here:



Solution: