ECE 102, Fall 2018 **Practice Problems**

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UCLA True Bruin academic integrity principles apply.
Open: Two pages of cheat sheet allowed.
Closed: Book, computer, Internet.

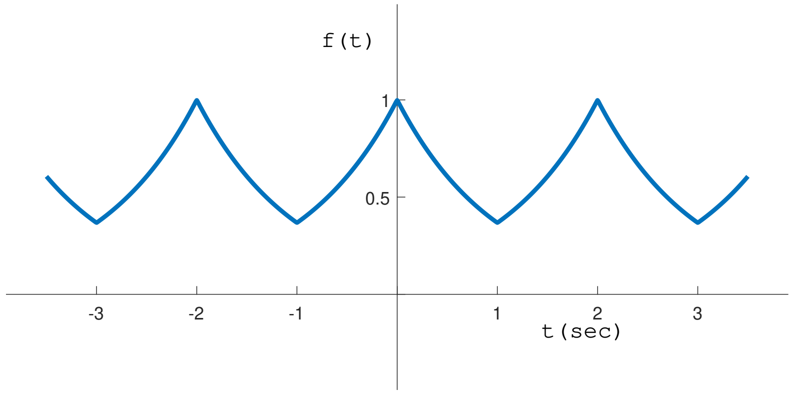
State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Name: ____________________

Signature: ____________________

ID#: _________________

1. $f(t)$ is a periodic signal with period $T_0 = 2$ s, where one period of the signal is defined as $e^{-|t|}$ for $-1 \leq t \leq 1$ s, as shown below.



(a) Find its Fourier series coefficients c_k.

(b) If we plot, using MATLAB, the truncated Fourier series $f_N(t) = \sum_{k=-N}^{N} c_k e^{j\frac{2\pi}{T_0}kt}$, will Gibbs phenomenon occur for this signal? Explain your answer.

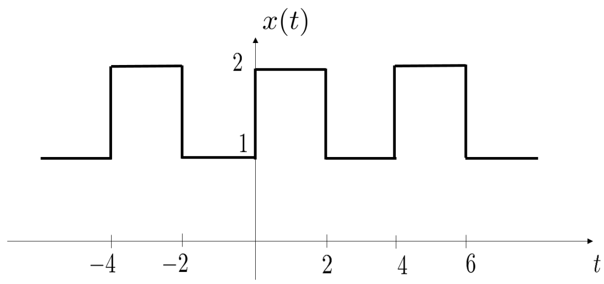
Solutions:

(a) The Fourier series coefficients of $f(t)$ are given by:

$$
\begin{aligned}
c_k &= \frac{1}{T_0}\int_{-1}^{1} f(t)e^{-j\omega_0 kt}dt \\
&= \frac{1}{2}\left(\int_{-1}^{0} e^{t}e^{-j\pi kt}dt + \int_{0}^{1} e^{-t}e^{-j\pi kt}dt\right) \\
&= \frac{1}{2}\left(\int_{-1}^{0} e^{(-j\pi k+1)t}dt + \int_{0}^{1} e^{(-j\pi k-1)t}dt\right) \\
&= \frac{1}{2}\left(\int_{-1}^{0} e^{(-j\pi k+1)t}dt + \int_{0}^{1} e^{(-j\pi k-1)t}dt\right) \\
&= \frac{1}{2}\left(\frac{1-e^{(j\pi k-1)}}{-j\pi k+1} + \frac{e^{(-j\pi k-1)}-1}{-j\pi k-1}\right) \\
&= \left(1-e^{-1}(-1)^k\right)\frac{1}{(1+\pi^2 k^2)}
\end{aligned}
$$

(b) The function $f(t)$ is continuous, there are no discontinuity points, therefore there will be no ripples when plotting $f_N(t)$. The Gibbs phenomenon happened when we had discontinues function.

2. Suppose we have a periodic signal $x(t)$, with period of $T_0 = 4$, and let a_k denote the Fourier series coefficients of $x(t)$. Suppose from $x(t)$, we construct a new signal, $y(t)$, that has the same period of $x(t)$. The Fourier series coefficients of $y(t)$ are given by: $b_k = (-1)^k a_k$. Express $y(t)$ in terms of $x(t)$ and sketch $y(t)$.

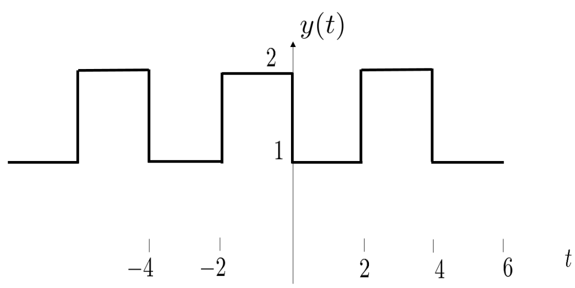


Solutions:

$$y(t) = \sum_{-\infty}^{\infty} b_k e^{j\omega_0 kt} = \sum_{-\infty}^{\infty} b_k e^{j\frac{\pi}{2}kt} = \sum_{-\infty}^{\infty} (-1)^k a_k e^{j\frac{\pi}{2}kt} = \sum_{-\infty}^{\infty} e^{j\pi k} a_k e^{j\frac{\pi}{2}kt} = \sum_{-\infty}^{\infty} a_k e^{j\frac{\pi}{2}k(t+2)}$$

Therefore,

$$y(t) = x(t+2)$$



3. Find the value of A in $x(t) = A\delta(t) - \text{sinc}(t)$ such that $x(t) * x(t) = x(t)$

Solutions:

$$\begin{aligned} x(t) * x(t) &= (A\delta(t) - \text{sinc}(t)) * (A\delta(t) - \text{sinc}(t)) \\ &= A^2\delta(t) - 2A\text{sinc}(t) + \text{sinc}(t) * \text{sinc}(t) \end{aligned}$$

Now,

$$\text{sinc}(t) * \text{sinc}(t) \to \text{rect}(\omega/2\pi)\text{rect}(\omega/2\pi) = \text{rect}(\omega/2\pi)$$

Therefore,

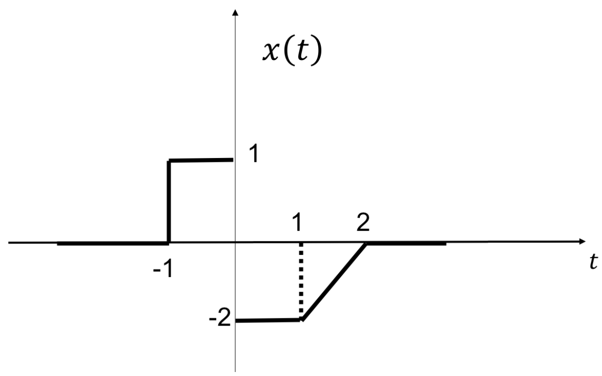
$$\text{sinc}(t) * \text{sinc}(t) = \text{sinc}(t)$$

Now,

$$x(t) * x(t) = A^2\delta(t) - 2A\text{sinc}(t) + \text{sinc}(t)$$

For $x(t) * x(t) = x(t)$, A should be 1.

4. Consider an LTI system with impulse response $h(t) = e^{-t}\delta(t) + u(t-1)$. We give this system the following input:



Let $y(t)$ denote its corresponding output. Find $y(t)$ at times $t = \frac{3}{2}$, $t = +\infty$.

Solutions:

We can first simplify $h(t)$ to the following: $h(t) = \delta(t) + u(t-1)$. Therefore,

$$\begin{aligned} y(t) &= x(t) * h(t) = x(t) * (\delta(t) + u(t-1)) \\ &= x(t) + \int_{-\infty}^{\infty} x(\tau)u(t-1-\tau)d\tau \\ &= x(t) + \int_{-\infty}^{t-1} x(\tau)d\tau \end{aligned}$$

Therefore,

$$\begin{aligned} y(3/2) &= x(3/2) + \int_{-\infty}^{0.5} x(\tau)d\tau = -1 + 1 - 2 * 0.5 = -1 \\ y(t)_{t\to\infty} &= 0 + \int_{-\infty}^{\infty} x(\tau)d\tau = 1 - 2 - 2 * 1/2 = -2 \end{aligned}$$

5. Show if each of the following systems is LTI. In the case where the system is LTI, determine its impulse response.

(a) $y(t) = \int_{-\infty}^{t} \lambda^{-(t-\tau)} x(\tau) d\tau$, where $\lambda \geq 1$

Solutions:

Suppose that for inputs $x_1(t)$ and $x_2(t)$, we have respectively the corresponding outputs $y_1(t)$ and $y_2(t)$ outputs. Now, let $x(t) = ax_1(t) + bx_2(t)$, we then have the following:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{t} \lambda^{-(t-\tau)} x(\tau) d\tau \\
&= \int_{-\infty}^{t} \lambda^{-(t-\tau)} (ax_1(\tau) + bx_2(\tau)) d\tau \\
&= \int_{-\infty}^{t} (a\lambda^{-(t-\tau)} x_1(\tau) + b\lambda^{-(t-\tau)} x_2(\tau)) d\tau \\
&= \int_{-\infty}^{t} a\lambda^{-(t-\tau)} x_1(\tau) d\tau + b\lambda^{-(t-\tau)} x_2(\tau) d\tau \\
&= \int_{-\infty}^{t} a\lambda^{-(t-\tau)} x_1(\tau) d\tau + \int_{-\infty}^{t} b\lambda^{-(t-\tau)} x_2(\tau) d\tau \\
&= ay_1(t) + by_2(t)
\end{aligned}$$

Therefore, the system is linear.

Time-invariance:

If we delay the input for t_0:

$$\begin{aligned}
y_{t_0}(t) &= \int_{-\infty}^{t} \lambda^{-(t-\tau)} x(\tau - t_0) d\tau, \quad \text{let } \tau' = \tau - t_0 \\
&= \int_{-\infty}^{t-t_0} \lambda^{-(t-\tau'-t_0)} x(\tau') d\tau' \\
&= y(t - t_0)
\end{aligned}$$

Therefore, the system is time-invariant. Now determining the impulse response:

$$h(t) = y(t)|_{x(t)=\delta(t)} = \int_{-\infty}^{t} \lambda^{-(t-\tau)} \delta(\tau) d\tau = \int_{-\infty}^{t} \lambda^{-t} \delta(\tau) d\tau = \lambda^{-t} \int_{-\infty}^{t} \delta(\tau) d\tau = \lambda^{-t} u(t)$$

(b) $y(t) = \begin{cases} x(t), & |x(t)| \leq 1 \\ 1, & x(t) > 1 \\ -1, & x(t) < -1 \end{cases}$

Solutions:

The system is not linear, we can check the homogeneity property: Let $x(t) = 0.5$, then $y(t) = 0.5$. Now if we give the system the following input $3x(t) = 1.5$, the output is then $1 \neq 3y(t)$. Therefore, the system is not linear.

The system is time-invariant. This is because if we delay the input by t_0: $x_{t_0}(t) = x(t - t_0)$, the corresponding output:

$$y_{t_0}(t) = \begin{cases} x_{t_0}(t), & |x_{t_0}(t)| \leq 1 \\ 1, & x_{t_0}(t) > 1 \\ -1, & x_{t_0}(t) < -1 \end{cases}$$

Therefore,

$$y_{t_0}(t) = \begin{cases} x(t-t_0), & |x(t-t_0)| \leq 1 \\ 1, & x(t-t_0) > 1 \\ -1, & x(t-t_0) < -1 \end{cases}$$

Since $y_{t_0}(t) = y(t-t_0)$, the system is time-invariant.

6. Evaluate the following integral:
$\int_{-\infty}^{\infty} \text{sinc}(2\tau + 1) d\tau$
Solutions:
Let $x(t) = \text{sinc}(2t+1)$. Then using the definition of Fourier transform, we have:

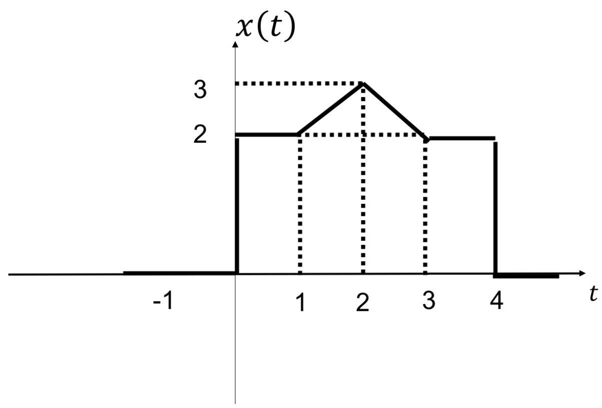
$$\int_{-\infty}^{\infty} x(\tau) d\tau = X(j\omega)|_{\omega=0}$$

Now,

$$X(j\omega) = \frac{1}{2}\text{rect}(\omega/4\pi)e^{j\omega/2}$$

Therefore, $X(0) = \frac{1}{2}$.

7. Consider the following real signal $x(t)$:



Let $X(j\omega)$ denote its Fourier transform. Evaluate the following:

(a) $\int_{-\infty}^{+\infty} X(j\omega)e^{-j\omega} d\omega$
Solutions:

$\int_{-\infty}^{+\infty} X(j\omega)e^{-j\omega} d\omega = 2\pi x(t)_{t=-1} = 0$

(b) $\int_{-\infty}^{+\infty} X(j(\omega-1))e^{2j\omega} d\omega$
Solutions:

$\int_{-\infty}^{+\infty} X(j(\omega-1))e^{2j\omega} d\omega = \int_{-\infty}^{+\infty} X(j\omega')e^{2j(\omega'+1)} d\omega' = e^{j2} \int_{-\infty}^{+\infty} X(j\omega')e^{2j\omega'} d\omega' = 2\pi e^{j2} x(t)_{t=2} =$
$6\pi e^{2j}$

(c) $\int_{-\infty}^{+\infty} \mathcal{R}e\{X(j\omega)\}e^{-j\omega}d\omega$

Solutions:
Since $x(t)$ is real, $Re\{X(j\omega)\} = X_e(j\omega)$.Therefore,

$$\int_{-\infty}^{+\infty} \mathcal{R}e\{X(j\omega)\}e^{-j\omega}d\omega = \int_{-\infty}^{+\infty} X_e(j\omega)e^{-j\omega}d\omega = 2\pi x_e(t)|_{t=-1} = 2\pi(x(1)+x(-1))/2 = 2\pi$$

8. Use Parseval's theorem to prove the following:

$$\text{Power of}\left(\sum_{k=0}^{\infty} A_k\cos(k\omega_0 t+\theta)\right) = |A_0\cos(\theta)|^2 + \sum_{k=1}^{\infty}\frac{1}{2}|A_k|^2$$

Solutions:

$$\begin{aligned}\sum_{k=0}^{\infty} A_k\cos(k\omega_0 t+\theta) &= A_0\cos(\theta) + \sum_{k=1}^{\infty} A_k\frac{1}{2}\left(e^{j(k\omega_0 t+\theta)} + e^{-j(k\omega_0 t+\theta)}\right)\\ &= A_0\cos(\theta) + \sum_{k=1}^{\infty}\frac{A_k}{2}e^{j\theta}e^{jk\omega_0 t} + \sum_{k=1}^{\infty}\frac{A_k}{2}e^{-j\theta}e^{-jk\omega_0 t}\end{aligned}$$

Therefore, the power is as follow:

$$|c_k|^2 = |A_0\cos(\theta)|^2 + \sum_{k=1}^{\infty}\frac{|A_k|^2}{4} + \sum_{k=1}^{\infty}\frac{|A_k|^2}{4} = |A_0\cos(\theta)|^2 + \sum_{k=1}^{\infty}\frac{|A_k|^2}{2}$$